SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

F17CC

Introduction to University Mathematics

Semester 1 – 2018/19

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

F17CC *Each question part is worth 5 marks.*

1. (a) A set X contains 15 elements. How many subsets does it have?

   (b) Write the complex number $\frac{1}{6+4i}$ in the form $a + bi$ where a and b are real numbers.

   (c) Carry out the following matrix multiplication

$$\begin{pmatrix} 1 & 4 & -1 \\ 2 & 0 & 3 \\ -5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & 3 \\ 3 & 4 & 1 \end{pmatrix}$$

   (d) Find the angle to the nearest degree between the vectors **a** and **b** where

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

2. (a) How many committees of 5 people can be formed from 120 bureaucrats?

   (b) Find the square roots of $-16 - 30i$ and show that your solutions work.

   (c) Solve the following system of linear equations. You **must** use elementary row operations. Show that your solutions work.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ -3x + 2y - 2z &= -10. \end{aligned}$$

   (d) Calculate the vector product $\mathbf{a} \times \mathbf{b}$ of the vectors **a** and **b** where

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Exam paper continues ...

3. (a) Factorize $x^4 + x^2 + 1$ as a product of two real irreducible quadratic polynomials.

   (b) Find the 4th roots of 3.

   (c) Find the inverse of the matrix below **by first finding its adjugate**, and show that your answer works

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 2 \end{pmatrix}.$$

   (d) Find the Cartesian equation of the plane that passes through the point with position vector $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and has $\mathbf{d} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ as its norma1.

4. (a) Show that the set of solutions to the following system of linear equations

$$\begin{aligned} 2x + y - 3z &= 0 \\ 4x + 2y - 6z &= 0 \\ x - y + z &= 0 \end{aligned}$$

   forms a line through the origin.

   (b) Prove by induction that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

   (c) Prove that for complex numbers u and v we have that $|u||v| = |uv|$.

   (d) Prove Pythagoras' theorem using vectors.

End of paper

SOLUTIONS TO EXAM PAPER 2018

1. (a) 2^{15}.

   (b) $\frac{3}{26} - \frac{1}{13}i$.

   (c) The determinant in this case is -1. So the inverse is simply -1 times the adjugate. The inverse is

$$\begin{pmatrix} -1 & 9 & 9 \\ 13 & 14 & -1 \\ -4 & 21 & 30 \end{pmatrix}$$

   (d) $73°$.

2. (a) $\binom{120}{5}$.

   (b) $\pm(3 - 5i)$. One mark for showing that solutions work.

   (c) $x = 2$, $y = -1$, $z = 1$. One mark for showing that solutions work.

   (d) $5\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$.

3. (a) $(x^2 - x + 1)(x^2 + x + 1)$.

   (b) The 4th roots of unity are ± 1, $\pm i$. The principal 4th root of 3 is $\sqrt{3}$. Thus the 4th roots of 3 are: $\pm\sqrt{3}$, $\pm\sqrt{3}i$.

   (c)

$$\begin{pmatrix} 2 & 0 & -1 \\ 3 & -1 & -1 \\ -2 & 1 & 1 \end{pmatrix}$$

   One mark for showing that solution works.

   (d) $x + 2y + 3z = 6$.

4. (a) The solution set is the line through the origin with parametric equation $\lambda(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$.

   (b) Base case: formula works when $n = 1$. The induction step amounts to showing that $(n+1)^3 + \frac{1}{4}n^2(n+1)^2 = \frac{1}{4}(n+1)^2(n+2)^2$.

   (c) Let $u = a + bi$ and $v = c + di$. Then the proof amounts to showing that $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.

(d) Let the legs of the perpendicular be $\mathbf{a}$ and $\mathbf{b}$ and let the longest side be $\mathbf{c}$ where $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Therefore $\mathbf{a} + \mathbf{b} = -\mathbf{c}$. Take inner products of both sides to get $\mathbf{a}^2 + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b}^2 = \mathbf{c}^2$. But $\mathbf{a} \cdot \mathbf{b} = 0$, since triangle is right-angled. Now observe that $\mathbf{x}^2$ is just the length of $\mathbf{x}$ squared.