Linear equations

This is a version of part of Section 8.3.

Theory

Matrix multiplication is defined in such a way that systems of linear equations can be written concisely as

$$A\mathbf{x} = \mathbf{b}.$$

The *solution set* of such a matrix equation is the set of all allowable $\mathbf{x}$ that satisfy the equation.

Theorem [Fundamental theorem of linear equations] *We assume that the scalars are the rationals, the reals or the complexes. For a system of linear equations* $A\mathbf{x} = \mathbf{b}$ *exactly one of the following holds.*

(1) There are no solutions. We say the equations are inconsistent*.*
(2) There is exactly one solution. The equations are consistent*.*
(3) There are infinitely many solutions. The equations are consistent*.*

Examples

Here are two examples of systems of linear equations. **They are easy to solve because of the *shapes* of the equations.**

(1) The single linear equation

$$x + 2y + 3z = 1$$

Has infinitely many solutions. Putting $x = \lambda \in \mathbb{R}$ and $y = \mu \in \mathbb{R}$, the solution set is

$$\left\{ \begin{pmatrix} 1 - 2\mu - 3\lambda \\ \mu \\ \lambda \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$$

which can also be written as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$$

(2) The system of two linear equation

$$\begin{aligned} x + 2y + 3z &= 1 \\ y + 2z &= 1 \end{aligned}$$

Has infinitely many solutions. Put $z = \lambda \in \mathbb{R}$. Then $y = 1 - 2\lambda$ and $x = 1 - 2(1 - 2\lambda) - 3\lambda = -1 + \lambda$. The solution set is therefore

$$\left\{ \begin{pmatrix} -1 + \lambda \\ 1 - 2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

which can also be written as

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$$

Practice

We describe an algorithm that will take as input a system of linear equations and produce as output the following: if the system has no solutions it will tell us; on the other hand if it has solutions then it will determine them all.

A matrix is called a *row echelon matrix* or is said to be in *row echelon form* if it satisfies the following three conditions.

(1) Any zero rows are at the bottom of the matrix below all the non-zero rows.
(2) If there are non-zero rows then they begin with the number 1, called the *leading 1*. (This is convenient but not essential).
(3) In the column beneath a leading 1, all the entries are zero.

The following is a picture of a typical row echelon matrix where the asterisks can be any elements.

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Echelon means arranged in a step-like manner. The row echelon matrices are precisely those which have a good shape, and a system of linear equations that have this step-like pattern is easy to solve.

The following operations on a matrix are called *elementary row operations*.

(1) Multiply row i by a non-zero scalar λ. Denote this operation by $R_i \leftarrow \lambda R_i$. This means that the lefthand side is replaced by the righthand side.
(2) Interchange rows i and j. Denote this operation by $R_i \leftrightarrow R_j$.
(3) Add a multiple λ of row i to another row j. Denote this operation by $R_j \leftarrow R_j + \lambda R_i$.

Proposition *Applying elementary row operations to a system of linear equations does not change their solution set.*

Given a system of linear equations $A\mathbf{x} = \mathbf{b}$ the matrix $(A|\mathbf{b})$ is called the *augmented matrix*.

Theorem [Gaussian elimination] *This is an algorithm for solving a system $A\mathbf{x} = \mathbf{b}$ of linear equations.* In outline, *the algorithm runs as follows.*

(Step 1): *Form the augmented matrix $(A|\mathbf{b})$.*
(Step 2): *By using elementary row operations, convert $(A|\mathbf{b})$ into a row echelon matrix $(A'|\mathbf{b}')$.*
(Step 3): *Solve the equations obtained from $(A'|\mathbf{b}')$ by back-substitution. These are also the solutions to the original equations.*

Examples

(1) We show that the following system of equations is inconsistent.

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2. \end{aligned}$$

The first step is to write down the augmented matrix of the system.

$$\left(\begin{array}{rrr|r} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right).$$

Carry out the elementary row operations $R_2 \leftarrow R_2 - 3R_1$ and $R_3 \leftarrow R_3 - 5R_1$. This gives

$$\left(\begin{array}{rrr|r} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right).$$

Now carry out the elementary row operation $R_3 \leftarrow R_3 - R_2$ which yields

$$\left(\begin{array}{rrr|r} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-3} \end{array}\right).$$

There is no need to continue. The equation corresponding to the last line of the augmented matrix is $0x + 0y + 0z = -3$. Clearly, this equation has no solutions because it is zero on the left of the equals sign and non-zero on the right. It follows that the original set of equations has no solutions.

(2) We show that the following system of equations has exactly one solution, and we shall also check it.

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 2y + 4z &= 0 \\ 3x + 4y + 5z &= 2. \end{aligned}$$

We first write down the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right).$$

Carry out the elementary row operations $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 3R_1$ to get

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & -2 & -4 & -10 \end{array}\right).$$

Now carry out the elementary row operations $R_2 \leftarrow -\frac{1}{2}R_2$ and $R_3 \leftarrow -\frac{1}{2}R_3$ that yield

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \end{array}\right).$$

Finally, carry out the elementary row operation $R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

This is a row echelon matrix and there are no free variables. Write down the corresponding set of equations

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + z &= 4 \\ z &= 1. \end{aligned}$$

Solve by back-substitution to get $x = -5$, $y = 3$ and $z = 1$. We check that

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.$$

(3) We show that the following system of equations has infinitely many solutions, and we shall check them.

$$\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14. \end{aligned}$$

The augmented matrix for this system is

$$\left(\begin{array}{rrr|r} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right).$$

Carry out the following elementary row operations $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 4R_1$, $R_2 \leftarrow -\frac{1}{5}R_2$, $R_3 \leftarrow -\frac{1}{5}R_3$ and $R_3 \leftarrow R_3 - R_2$. This yields

$$\left(\begin{array}{rrr|r} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right).$$

Because the bottom row consists entirely of zeros, this means that there are only two equations

$$\begin{aligned} x + 2y - 3z &= 6 \\ y - 2z &= 2. \end{aligned}$$

The variable z can be assigned any value $z = \lambda$ where $\lambda \in \mathbb{R}$. By back-substitution, both x and y can be expressed in terms of λ. The solution set can be written in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

We now check that these solutions work

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2 - \lambda \\ 2 + 2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

as required. Observe that the λs cancel out.