HERIOT WATT UNIVERSITY

SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

F17CC

Introduction to University Mathematics

Semester 1 – 2016/17

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

1. *Each part of this question is worth 5 marks.*

   (a) A set X contains 12 elements.

   i. How many subsets does X have? [3 marks]

   ii. How many of those subsets contain exactly 8 elements? [2 marks]

   (b) Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{4, 5, 6\}$, $C = \{a, b, c, d\}$ and $D = \{a, d\}$.

   i. What is the cardinality of $A \times C$? [2 marks]

   ii. What is the set $(A \setminus B) \times (C \setminus D)$? [3 marks]

   (c) Write

$$\left(\frac{3+4i}{5+6i}\right)^2$$

   in the form $a + ib$ where a and b are real numbers.

   (d) Find both square roots of $33 + 56i$.

   (e) Find the roots of the quadratic $x^2 - 4x + 13$.

   (f) Write $x^3 - 6x^2 + 21x - 26$ as a product of real linear and real irreducible quadratic factors.

   (g) Calculate A^2 where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

(h) Find all solutions to the following system of real linear equations using *elementary row operations*

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 1 \\ 3x + 4y + 5z &= 3. \end{aligned}$$

and show that your solutions work.

(i) Find the adjugate and inverse of the following matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

Show that your inverse works.

(j) Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$. Calculate the angle between $\mathbf{a}$ and $\mathbf{b}$ to the nearest degree.

(k) Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$. Find a non-zero vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(l) Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = -3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ and $\mathbf{c} = 5\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$. Calculate the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Each of the following questions is worth 10 marks.

2. (a) Prove that there are infinitely many primes. [5 marks]

   (b) Prove that the sum of the angles in a triangle is 180°. Make clear any assumptions you make. [5 marks]

3. (a) Define what is meant by a *Pythagorean triple*. Let

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

be the unit circle in the plane with centre the origin. Define what is meant by a *rational point* on C. Give an example of an *irrational point* on C. Prove that every rational point on C arises from a Pythagorean triple. [5 marks]

(b) Prove that $\sqrt{p}$ is irrational for each prime p. You may use without proof the result that if $p \mid a^2$ then $p \mid a$. [5 marks]

4. (a) Prove that the complex number z is real if and only if $\bar{z} = z$. [2 marks]

(b) Prove that for complex numbers u and v we have that $|uv| = |u|\,|v|$. [3 marks]

(c) Prove that every non-zero complex number has n nth roots. [5 marks]

5. (a) Let A, B and C be $n \times n$ real matrices. Prove that $(AB)C = A(BC)$. [5 marks]

(b) Prove that if A and B are invertible $n \times n$ matrices then AB is invertible and determine its inverse in terms of the inverses of A and B. [5 marks]

End of paper

Solutions to 2016 exam

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(a) (i) 2^{12} (ii) $\binom{12}{8}$

(b) (i) 24 (ii) $\{1,2,3\} \times \{b,c\}$.

(c) $\dfrac{1517}{3721} + \dfrac{156}{3721} i$.

(d) Put $(x+iy)^2 = 33+56i$.

Leads to three equations ① $x^2 - y^2 = 33$ ② $2xy = 56$ ③ $x^2+y^2 = 65$. ①+③ gives $x^2 = 49$ and so $x = \pm 7$ (two values!) By ②, $y = \frac{28}{x}$. Thus when $x=7$ we have that $y=4$ and when $x=-7$ we have that $y=-4$. The required roots are therefore $\pm(7+4i)$. Solutions are checked by squaring thusly $(7+4i)^2 = 33+56i$.

(e) This is just a quadratic equation whose roots are $2+3i$ and $2-3i$ (Complex conjugates!)

(f) $(x-2)(x^2-4x+13)$ [uses (e)!]. Observe that the question asks for <u>real</u> linear and <u>real</u> <u>irreducible</u> quadratic factors. You should also distinguish between roots (which are numbers) and linear factors (which are linear polynomials). Though these two concepts are related they are different.

(g)
$$A^2 = \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \\ 102 & 126 & 150 \end{pmatrix}$$

[Don't even think about calculating A^2 by squaring each element of A.]

(h) Write down the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 4 & 5 & 3 \end{array}\right)$$

By means of elementary row operations, we obtain the following

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (*)$$

the fact that the last row consists entirely of zeros both before and after the dividing line means that the system is consistent and there are infinitely many solutions. The equations corresponding to $(*)$ are $x+y+z=1$ and $y+2z=0$. Put $z=\lambda\ (\in \mathbb{R})$. Then $y=-2\lambda$ and $x=1+\lambda$. The solution set is therefore 1-parameter

$$\left\{ \begin{pmatrix} 1+\lambda \\ -2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

We now need to check all the solutions. And I mean all.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1+\lambda \\ -2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 1+\lambda-2\lambda+\lambda = 1 \\ 1+\lambda-4\lambda+3\lambda = 1 \\ 3+3\lambda-8\lambda+5\lambda = 3 \end{pmatrix} \checkmark$$

(Call the given matrix A.)

(i) We calculate the adjugate in stages.

- $$\begin{pmatrix} \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & -2 & -8 \\ 0 & -3 & -6 \\ -1 & -1 & -1 \end{pmatrix}$$

- Pattern of signs

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

- This leads to the matrix

$$\begin{pmatrix} 1 & 2 & -8 \\ 0 & -3 & 6 \\ -1 & 1 & -1 \end{pmatrix}$$

- The adjugate is the transpose of the above matrix

$$\operatorname{adj} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \\ -8 & 6 & -1 \end{pmatrix}$$

<u>Check</u> $$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \\ -8 & 6 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

It follows that $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{vmatrix} = -3$ or this can be computed directly.

Finally, the inverse of the given matrix is given by

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \\ -8 & 6 & -1 \end{pmatrix}$$

My check above shows that my calculations are correct or show that $AA^{-1} = I$ directly.

[It is important that checks are genuine checks. If your check doesn't work then you are in error and you should go back and correct your calculations. Don't just go through the motions.]

(j) We have that $\underline{a}\cdot\underline{b} = \|\underline{a}\|\,\|\underline{b}\|\cos\theta$. Thus $\cos\theta = \dfrac{\underline{a}\cdot\underline{b}}{\|\underline{a}\|\,\|\underline{b}\|}$. For the given $\underline{a}$ and $\underline{b}$ we have that

$$\cos\theta = \frac{-3+8+30}{\sqrt{1+4+25}\sqrt{9+16+36}} = \frac{35}{\sqrt{30}\sqrt{61}} \simeq 0{\cdot}82$$

It follows that $\theta = 35^\circ$ to the nearest degree.

(k) A vector orthogonal to both $\underline{a}$ and $\underline{b}$ is $\underline{a} \times \underline{b}$. We calculate this as follows

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 5 \\ -3 & -4 & 6 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ -4 & 6 \end{vmatrix} \underline{i} - \begin{vmatrix} 1 & 5 \\ -3 & 6 \end{vmatrix} \underline{j} + \begin{vmatrix} 1 & -2 \\ -3 & -4 \end{vmatrix} \underline{k}$$

$$= (-12+20)\underline{i} - (6+15)\underline{j} + (-4-6)\underline{k}$$

$$= 8\underline{i} - 21\underline{j} - 10\underline{k}.$$

Check (Always and without being asked)

$$(\underline{i} - 2\underline{j} + 5\underline{k}) \cdot (8\underline{i} - 21\underline{j} - 10\underline{k}) = 8 + 42 - 50 = 0 \checkmark$$

$$(-3\underline{i} - 4\underline{j} + 6\underline{k}) \cdot (8\underline{i} - 21\underline{j} - 10\underline{k}) = -24 + 84 - 60 = 0 \checkmark$$

(l) By definition $[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c})$

$= -[\underline{c}, \underline{b}, \underline{a}] = [\underline{c}, \underline{a}, \underline{b}]$. So, one way to compute this is to calculate

$$\underline{c} \cdot (\underline{a} \times \underline{b}) = (5\underline{i} + 7\underline{j} + 4\underline{k}) \cdot (8\underline{i} - 21\underline{j} - 10\underline{k})$$

$$= -147.$$

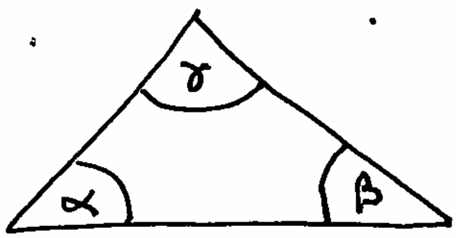
Alternatively,

$$[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} 1 & -2 & 5 \\ -3 & -4 & 6 \\ 5 & 7 & 4 \end{vmatrix} = -147.$$

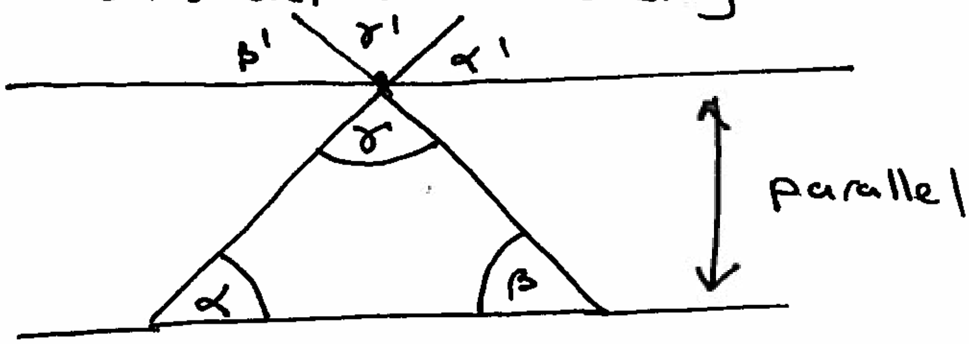
All of the following proofs were discussed in the lectures and exercises so I shall only sketch them out below.

2 (a). Show first that the smallest non-trivial ($\neq 1$) divisor of a number (≥ 2) is a prime. Now for the main result. Let $p_1, \ldots, p_n$ be the first n primes. Put $N = (p_1 \ldots p_n) + 1$. Either N is a prime $> p_n$ or N has a prime divisor $> p_n$. In either event, we can always find a bigger prime.

(b) Given triangle

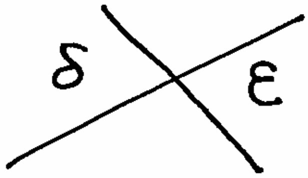


Now construct the following



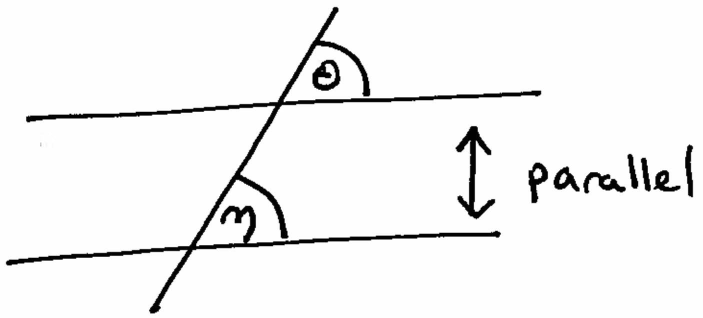
This uses the result that given a line l and a point P not on l, we can construct a unique line l' parallel to l and containing P.

We now need the following two results



$\Rightarrow \quad \delta = \varepsilon$.

and



$\Rightarrow \quad \eta = \theta$.

Using these, we deduce that $\gamma = \gamma'$ and $\beta = \beta'$ and $\alpha = \alpha'$. But clearly, $\alpha + \beta + \gamma$ is a straight line and so $180°$.

3 (a). A triple $(a, b, c) \in \mathbb{Z}^3$ is called a <u>Pythagorean triple</u> if $a^2 + b^2 = c^2$.

The point $(r, s) \in C$ is said to be <u>rational</u> if both $r, s \in \mathbb{Q}$.

An example of an <u>irrational</u> point on C is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Let $\left(\frac{a}{b}, \frac{c}{d}\right)$ be a rational point on C.

Then $\dfrac{a^2}{b^2} + \dfrac{c^2}{d^2} = 1$. It follows that

$a^2d^2 + c^2b^2 = b^2d^2$. Thus

(ad, bc, bd) is a Pythagorean triple.

Now let (p, q, r) be a Pythagorean triple (where $r \neq 0$). Then $p^2 + q^2 = r^2$ and so

$\left(\frac{p}{r}\right)^2 + \left(\frac{q}{r}\right)^2 = 1$. It follows that $\left(\frac{p}{r}, \frac{q}{r}\right)$

is a rational point on C.

Suppose now that (ad, bc, bd) (where $bd \neq 0$) is a Pythagorean triple then $a^2d^2 + b^2c^2 = b^2d^2$ and so $\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 1$.

It follows that our original rational point $\left(\frac{a}{b}, \frac{c}{d}\right)$ on C arises from a Pythagorean triple.

(b). Suppose that $\sqrt{p}$ is rational.

Then we can write $\sqrt{p} = \frac{a}{b}$ where $\frac{a}{b}$ is a fraction in its lowest terms (this is important).

Squaring and rearranging, we get $pb^2 = a^2$.

It follows that $p \mid a^2$. Using the result stated in the question, we deduce that $p \mid a$. We can therefore write $a = px$ for some x.

Substituting $a = px$ into $pb^2 = a^2$ we get $pb^2 = a = (px)^2 = p^2 x^2$. Cancelling p, we arrive at $b^2 = px^2$. It follows that $p \mid b^2$.
Once again, we use the result stated in the question to deduce that $p \mid b$.

We have therefore shown that $p \mid a$ and $p \mid b$.
But this contradicts the assumption that $\frac{a}{b}$ is a fraction in its lowest terms.
We deduce that $\sqrt{p}$ cannot be rational and so $\sqrt{p}$ is irrational,

4 (a). If z is real then $z = a + 0i$. By definition, $\bar{z} = a - 0i$ but this is just z. Thus z real $\Rightarrow z = \bar{z}$.
Now suppose that $z \in \mathbb{C}$ such that $z = \bar{z}$. Let $z = a + ib$. Then $\bar{z} = a - ib$ and so $a + ib = a - ib$.
Equating real and imaginary parts, we deduce that $b = -b$ and so $b = 0$. It follows that z is real.
Thus $z = \bar{z} \Rightarrow z$ is real.
We have therefore proved that z is real $\Leftrightarrow z = \bar{z}$.

4 (b). Put $u = a+ib$ and $v = c+id$.
Then $uv = (ac-bd) + i(ad+bc)$.

By definition,

$$|uv| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$$
$$= \sqrt{(ac)^2 - 2(ac)(bd) + (bd)^2 + (ad)^2 + 2(ad)(bc) + (bc)^2}$$
$$= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}.$$

Now

$$|u||v| = \sqrt{(a^2+b^2)(c^2+d^2)}$$
$$= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}.$$

It follows that $|uv| = |u||v|$.

4(c). Show first that unity has n n^{th} roots

Put $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Then the n^{th} roots of unity are

$1\ (=\omega^0),\ \omega, \omega^2, \ldots, \omega^{n-1}$

This follows from the observation ~~to~~ that

$(\omega^r)^n = (\omega^n)^r = 1^r = \underline{1}$ using de Moivre.

Now let $z = r(\cos\theta + i\sin\theta)$ (where $z \neq 0$).

(If $\theta = 0$ then replace by 2π).

Put $u = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$.

Then $u^n = z$. The n^{th} roots of u are therefore $u, \omega u, \ldots, \omega^{n-1}u$.

[The whole point of this result is that you do not need to apply the fundamental theorem of algebra (whose proof is outside the scope of this course) but use instead the geometric properties of the complex numbers]

5(a) Clearly, $(AB)C$ and $A(BC)$ have the same size, so it remains to show that

$$((AB)C)_{ij} = (A(BC))_{ij}.$$

By definition,

$$((AB)C)_{ij} = \sum_{k=1}^{n} (AB)_{ik} C_{kj}$$

$$= \sum_{k=1}^{n} \left(\sum_{l=1}^{n} A_{il} B_{lk} \right) C_{kj}.$$

We now use distributivity ~~[illegible]~~ in $\mathbb{R}$ to get

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} A_{il} B_{lk} C_{kj}$$

By commutativity, we may interchange summations and, working backwards we get

$$= (A(BC))_{ij}.$$

5(b). By assumption, A and B are invertible and so both A^{-1} and B^{-1} exist.

We calculate first

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$$

$$= AA^{-1} = I$$

where we use associativity and the definition of inverses.

Similarly, $(B^{-1}A^{-1})AB = I$.

It follows that AB is invertible and, by the uniqueness of inverses, $(AB)^{-1} = B^{-1}A^{-1}$.