HERIOT WATT UNIVERSITY

SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

+ SOLUTIONS

F17CC

Introduction to University Mathematics

Semester 1 – 2017/18

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

F17CC *Each question part is worth 5 marks.*

1. (a) A set X contains 10 elements. How many subsets of X are there that contain either 2 or 3 elements?

   (b) A *squawk* is a message 153 characters long. A character is either a space or one of the 26 upper-case Latin letters. How many squawks are there?

   (c) Find the coefficient of $x^{10}y^{20}$ in the binomial expansion of

   $$(3x - 5y)^{30}.$$

   (d) Prove that

   $$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

2. (a) Write $(5 + 9i)^{-1}$ in the form $a + bi$ where $a, b \in \mathbb{R}$.

   (b) Find the square roots of $-21 + 220i$ and show that your solutions work.

   (c) Solve the following system of linear equations if this is possible. You **must** use elementary row operations.

   $$\begin{aligned} x + 2y + z &= 3 \\ 2x + 5y + 2z &= 4 \\ 2x + 4y + z &= 5. \end{aligned}$$

   (d) Solve the following system of linear equations if this is possible. You **must** use elementary row operations.

   $$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + z &= 3 \\ 3x + 3y + 2z &= 4. \end{aligned}$$

Exam paper continues ...

3. Let A be the following matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 3 & 0 \\ -1 & -2 & 5 \end{pmatrix}.$$

(a) Calculate the determinant of A.

(b) Calculate the inverse of A using the adjugate of A.

(c) Calculate the characteristic polynomial of A.

(d) Find the eigenvalues of A.

4. (a) Find the angle to the nearest degree between the vectors $\mathbf{a}$ and $\mathbf{b}$ below

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}.$$

(b) Find the vector product of the the vectors $\mathbf{a}$ and $\mathbf{b}$ below

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}.$$

(c) Find the Cartesian equations of the line which passes through the point with position vector $5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and is *parallel to* the vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

(d) Find the Cartesian equation of the plane through the point with position vector $7\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ and with normal $3\mathbf{i} + 6\mathbf{j} + \mathbf{k}$.

End of paper

SOLUTIONS TO EXAM PAPER 2017

1. (a) $\binom{10}{2}+\binom{10}{3}$.
   (b) 27^{153} squawks.
   (c) The binomial expansion is
   $$\sum_{i=0}^{30}\binom{30}{i}(3x)^i(-5y)^{30-i}.$$
   Put $i=10$. The coefficient is
   $$\binom{30}{10}3^{10}5^{20}.$$
   (d) There are two ways to prove this. (1) By the binomial expansion of $(x+y)^n$ and putting $x=y=1$. (2) By counting: the total number of subsets of an n-element set is 2^n; the number of k-subsets of an n-element set is $\binom{n}{k}$.
2. (a) $\frac{1}{106}(5-9i)$.
   (b) $\pm(10+11i)$.
   (c) There is a unique solution $x=6, y=-2, z=1$.
   (d) There are infinitely many solutions
   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-\mu \\ \mu \\ -1 \end{pmatrix}.$$
3. (a) 36.
   (b) The inverse is
   $$\frac{1}{36}\begin{pmatrix} 15 & -3 & 3 \\ -10 & 14 & -2 \\ -1 & 5 & 7 \end{pmatrix}.$$
   (c) $-x^3+11x^2-36x+36$.
   (d) The eigenvalues are $2, 3, 6$, the roots of the characteristic polynomial.

4. (a) 138°.

   (b) $-6\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$.

   (c) First, write down the vector equation of this line

$$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}).$$

   Next, write out in co-ordinate form

$$x = 5 + 2\lambda, \quad y = -2 - \lambda, \quad z = 4 + 3\lambda.$$

   The Cartesian equations result by eliminating λ

$$\begin{aligned} x + 2y &= 1 \\ 3y + z &= -2. \end{aligned}$$

   (d) If $\mathbf{a}$ is the point and $\mathbf{n}$ is the normal, the desired Cartesian equation is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

   Thus the required equation is

$$3x + 6y + z = 0.$$