Solutions: Exercises 1

1.

	vertices	edges
(a)	4	5
(b)	7	8
(c)	10	22

(a) $\deg(a) = 2$, $\deg(b) = 1$, $\deg(c) = 3$, $\deg(d) = 4$.

(b) $\deg(a) = 2$, $\deg(b) = 2$, $\deg(c) = 3$, $\deg(d) = 2$, $\deg(e) = 4$, $\deg(f) = 2$, $\deg(g) = 1$.

(c) $\deg(1) = 5$, $\deg(2) = 5$, $\deg(3) = 4$, $\deg(4) = 4$, $\deg(5) = 4$, $\deg(6) = 5$, $\deg(7) = 4$, $\deg(8) = 4$, $\deg(9) = 4$, $\deg(10) = 5$.

In each case, you should find that the sum of the vertex degrees is twice the number of edges. This is just the 'Handshaking Lemma'.

2. (a) The last one.

   (b) The middle one.

   (c) The last one.

3. (a) (1), (5).

   (b) (3).

   (c) (2), (4).

   (d) (1), (2), (3).

   (e) (4).

   (1) $(1, 1, 1, 1, 4, 4)$.

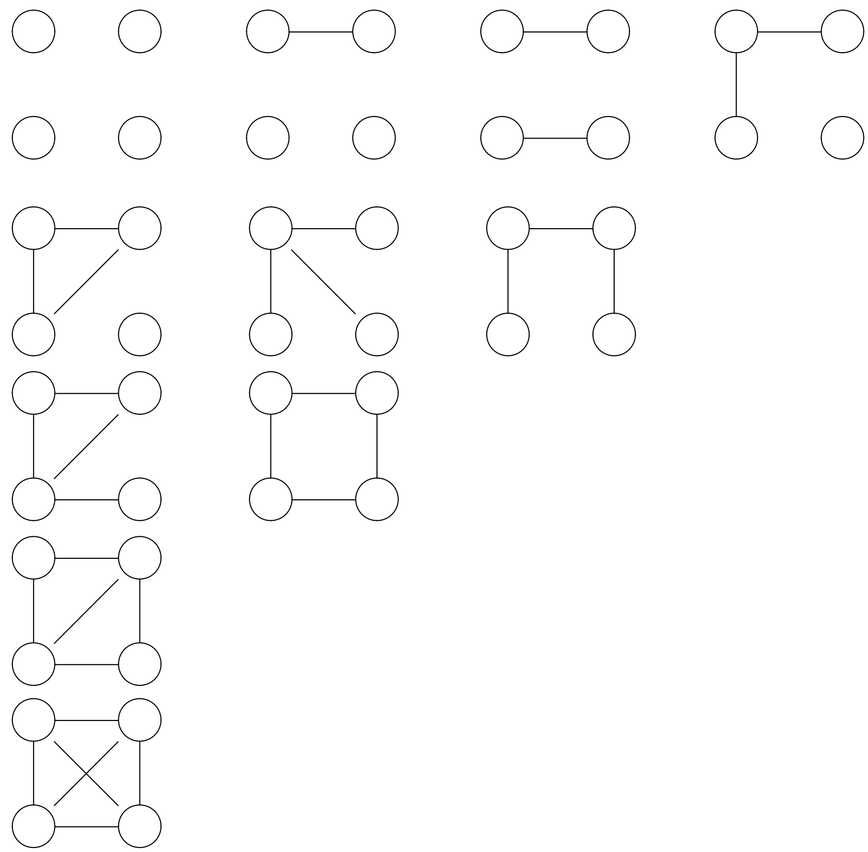
   (2) $(4, 4, 4, 4, 4)$.

   (3) $(1, 1, 3, 3)$.

   (4) $(0, 0, 0, 0)$.

   (5) $(1, 1, 2, 2, 2, 2, 2, 4)$.

4. There are 11 such graphs.

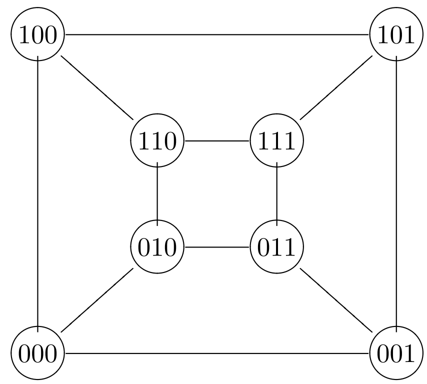


5. A cycle of length 5 is abcdea; a cycle of length 6 is afhcdea; a cycle of length 8 is abchjgifa; a cycle of length 9 is abcdigjhfa.

6. The null graph N_n has n vertices and 0 edges. The cycle graph C_n has n vertices and n edges. The complete graph K_n has n vertices and $\frac{1}{2}n(n-1)$ edges. This can be proved by observing that the number of edges is equal to the number of 2 element subsets of an n element set. Alternatively, we can use the Handshaking Lemma: the graph is regular of degree $n-1$. Thus the sum of the vertex degrees is $n(n-1)$. The number of edges is half this number. The complete bipartite graph $K_{m,n}$ has $m+n$ vertices and mn edges.

7. Here is the graph obtained for 3 bits. The one for 4 bits is similar but

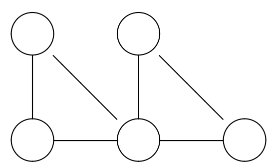
more complicated.



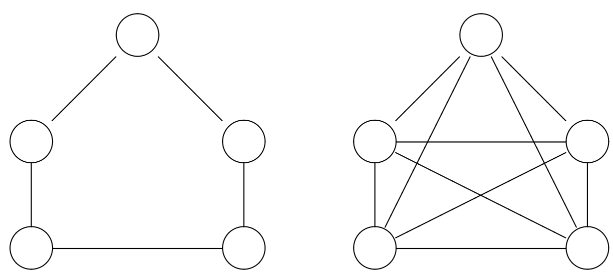
8. The Fibonacci numbers.

9. In each case, there will usually be more than one solution.

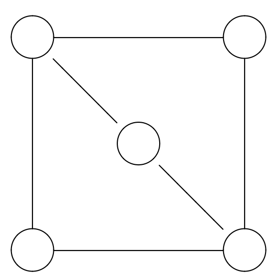
(a)



(b)

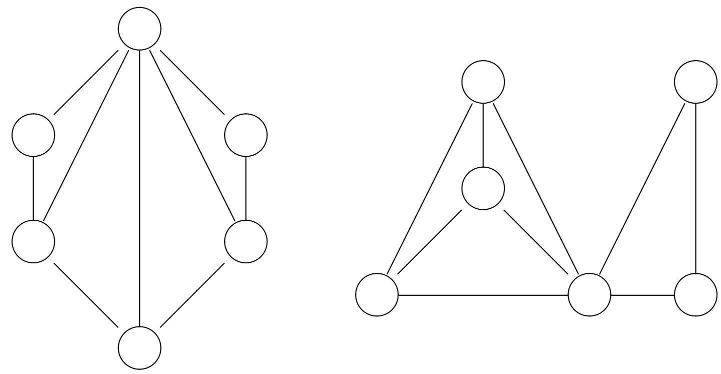


(c)



(d) Impossible: the sum of the vertex degrees must be even by the Handshaking Lemma.

(e)



10. By the Handshaking Lemma, the sum of the vertex degrees is twice the number of edges. In particular, the sum of the vertex degrees is even. The sum of n odd numbers is even iff n is even, whereas the sum of even numbers is always even. Thus the number of vertices of odd degree is even.

11. In a simple graph with n vertices, each edge is joined to at most $n - 1$ others.

    Suppose that there is a vertex of degree $n - 1$. Then every vertex has degree at least 1. Thus the possible vertex degrees lie between 1 and $n - 1$. But there are n vertices and so there must be at least two vertices with the same degree.

    Suppose that there is no vertex with the maximum vertex degree $n - 1$. Then the possible vertex degrees are 0 to $n - 2$. Once again there are $n - 1$ possible vertex degrees but n vertices and so at least two vertices have the same degree.

    Both arguments use the *pigeon-hole principle*.