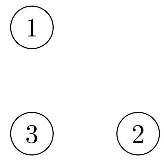
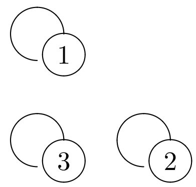
Exercises 2

1. For each of the graphs below, write down the adjacency matrix.

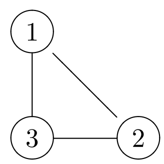
   (i)



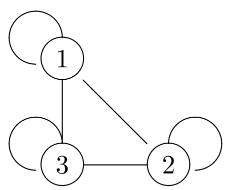
   (ii)



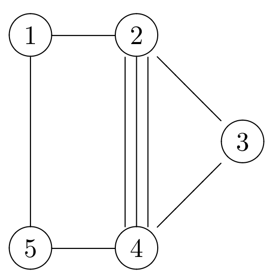
   (iii)



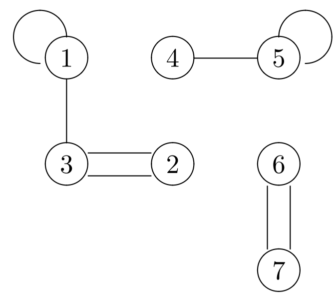
   (iv)



   (v)



(vi)



2. Draw the graphs having the following adjacency matrices.

   (i)

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   (ii)

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

   (iii)

$$\begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{pmatrix}$$

3. For each of the graphs in Question 1, find the matrices giving the paths of length 3 between any two vertices.

4. What do the adjacency matrices of the null graphs N_n look like?

5. What do the adjacency matrices of the cyclic graphs C_n look like?

6. What do the adjacency matrices of the complete graphs K_n look like?

7. What do the adjacency matrices of the complete bipartite graphs $K_{m,n}$ look like?

8. When is a matrix the adjacency matrix of a graph? Deduce that if A is the adjacency matrix of a graph so is A^n for all natural numbers $n \geq 1$.

9. Let G be a *simple* graph with adjacency matrix A.

   (i) What can you say about the diagonal entries of A?

   (ii) What can you say about the sum of the diagonal entries of A^2?

(iii) What can you say about the sum of the diagonal entries of A^3?

10. Let A be the adjacency matrix of K_5. Let n be a positive integer. Explain why all the diagonal entries of A^n are equal, and all the off-diagonal elements are equal. Let d_n be the common value of the diagonal elements of A^n and let a_n be the common value of the off-diagonal elements of A^n. What are a_1 and d_1, and a_2 and d_2? Show that:

$$d_{n+1} = 4a_n, \quad a_{n+1} = d_n + 3a_n, \quad a_{n+1} = 3a_n + 4a_{n-1}.$$

11. Prove the following by induction.

    (i) $n^3 + 2n$ is exactly divisible by 3 for all natural numbers $n \geq 1$.

    (ii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all natural numbers $n \geq 1$.

    (iii) $n! \geq 2^{n-1}$ for all natural numbers $n \geq 1$.

12. Prove for all $n \geq 1$ that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

13. Prove for all $n \geq 0$ that the following number is exactly divisible by 17

$$3 \cdot 5^{2n+1} + 2^{3n+1}.$$

14. A matrix A is said to be *symmetric* if it is equal to its transpose; that is, $A^T = A$. Prove that if A is symmetric then A^n is symmetric for all $n \geq 1$.

15. Prove that $n^3 < 3^n$ for all $n \geq 4$.