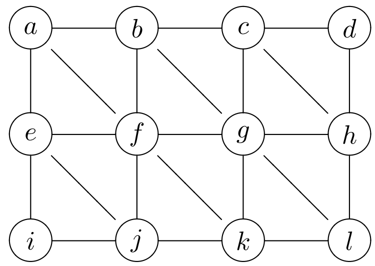
Exercises 3

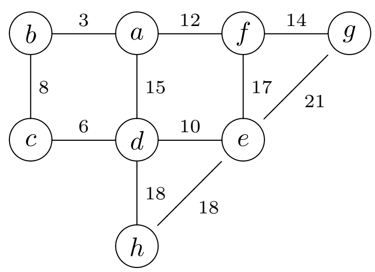
1. Draw all (unlabelled) trees with at most 6 vertices.

2. Consider the following graph.



The vertices represent offices and two vertices are joined by an edge if there is a direct communication link between them. Even if some edges are destroyed communication between any two offices may still be possible by relaying messages through intermediate offices. Find the maximum number of edges that can be destroyed whilst still ensuring that any two offices can communicate, and justify your answer.

3. Apply Prim's algorithm to the weighted graph drawn below showing all steps in the application of the algorithm. You should also write down the total weight of your minimum spanning trees.



4. The following table shows the distances (in miles) between six places in Scotland:

	Aberdeen	Edinburgh	Fort William	Glasgow	Inverness	Perth
Aberdeen	-	120	147	142	104	81
Edinburgh	120	-	132	42	157	45
Fort William	147	132	-	102	66	105
Glasgow	142	42	102	-	168	61
Inverness	104	157	66	168	-	112
Perth	81	45	105	61	112	-

Represent this information by means of a weighted graph. Use Prim's algorithm to find a minimum spanning tree — what is the overall total weight of this tree?

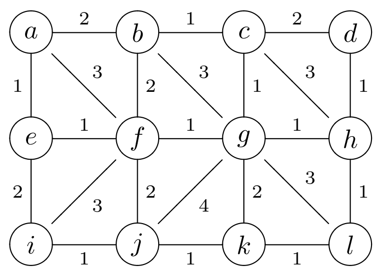
5. How many spanning trees does K_4 have?

6. In what way is the algorithm **spanning tree** a special case of Prim's algorithm?

7. Prove that a tree, with at least 2 vertices, contains at least two vertices of degree 1.

8. An *alkane* is a tree with two kinds of vertices: those labelled C have degree 4 and those labelled H have degree 1. These are the only kinds of vertices. Draw an alkane with three vertices of type C. If there are n vertices labelled C prove that there are $2n + 2$ vertices labelled H.

9. Apply Prim's algorithm to the weighted graph below, and write down the total weight of the minimum spanning tree obtained.



10. Prove each of the following using induction.

(i) Prove that $n^3 - n$ is exactly divisible by 3 for all integers $n \geq 0$.

(ii) Prove that $11^n - 6$ is exactly divisible by 5 for all integers $n \geq 1$.

(iii) Prove that $(1 + x)^n \geq 1 + nx$ for all non-negative real x and integers $n \geq 0$.

(iv) Prove that $2^n > n + 1$ for all integers $n \geq 2$.

(v) Find the smallest positive integer n for which it is true that $n! \geq 2^n$. Call this integer n_0. Prove for all $n \geq n_0$ that $n! \geq 2^n$.

(vi) Prove that $\sum_{i=1}^{n} 2i = n(n + 1)$.

(vii) Prove that $\sum_{i=1}^{n} (2i)^2 = \frac{2}{3}n(n + 1)(2n + 1)$.