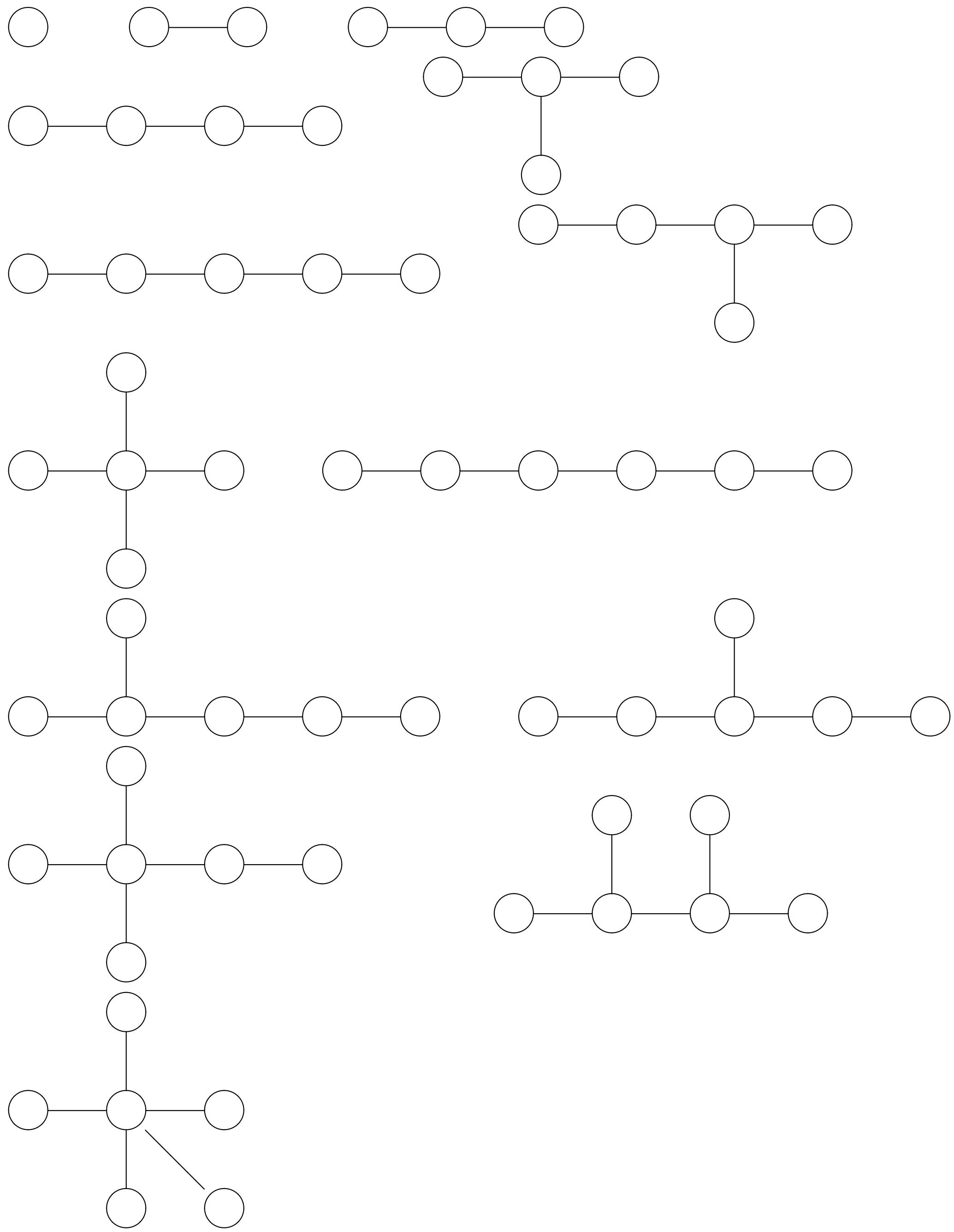
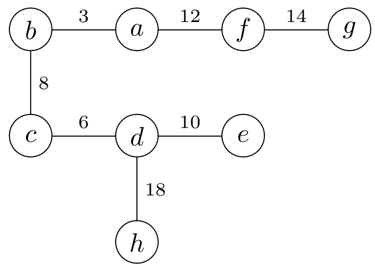
Solutions: Exercises 3

1. This is a question about *unlabelled* graphs which makes it a lot more tractable. There are 14 altogether.



2. There are 12 vertices and 23 edges. The smallest number of edges that a connected graph with 12 vertices can have is 11. Thus at most 12 edges can be destroyed so that the resulting graph remains connected and — meaning that any two offices can still communicate.

3.



The total weight of any minimum weight spanning tree is 71.

4. The graph you get will be K_6 with the vertices labelled by the cities and the edges labelled by the distances. The minimum weight spanning tree joins Aberdeen to Perth, Perth to Edinburgh, Edinburgh to Glasgow, Glasgow to Fort William, and Fort William to Inverness and has a total weight of 336 miles.

5. 16 — you just have to list them all. Note that when finding spanning trees we assume the graph is *labelled*.

6. The algorithm **spanning tree** is obtained from Prim's algorithm when the graph is weighted by placing unity on each edge.

7. A tree with at least 2 vertices cannot have any vertices of degree 0 because trees are by definition connected. Let the tree in question have n vertices and degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. By the above $d_1 \geq 1$. The number of edges is $n-1$ and so by the Handshaking Lemma we have that

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

Suppose that there are no vertices of degree 1. Then $d_1 \geq 2$. It follows that

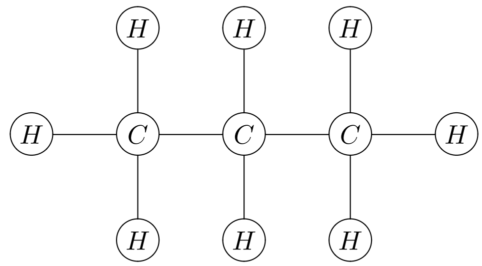
$$2(n-1) \geq 2n,$$

which is nonsense. Thus there are vertices of degree 1. Suppose that there is exactly one vertex of degree 1. Then $d_1 = 1$ and $d_2 \geq 2$. Thus

$$2(n-1) \geq 1 + 2(n-1),$$

which is nonsense. Thus there are at least two vertices of degree 1.

8. An alkene with three vertices of type C is



Suppose there are n vertices labelled C and m vertices labelled H. The total number of vertices is $n + m$ and so, since the graph is a tree, the number of edges is $n + m - 1$. By the Handshaking Lemma, we have that $4n + m = 2(n + m - 1)$. We have to find m in terms of n. This is just $2n + 2$.

9. The minimum weight spanning tree has weight 11.

10. In each case I have given a sketch.

(i) Observe that $(n + 1)^3 - (n + 1) = (n^3 - n) + 3(n^2 + n)$.

(ii) Observe that

$$11^{n+1} - 6 = 11 \cdot 11^n - 6 = (10 + 1)11^n - 6 = 10 \cdot 11^n + (11^n - 6).$$

(iii) Observe that

$$(1 + x)^{n+1} = (1 + x)(1 + x)^n \geq (1 + x)(1 + nx)$$

by the induction hypothesis because $(1 + x) \geq 0$. Now multiply out to get $1 + (1 + n)x + nx^2$. But $nx^2 \geq 0$ and so we get $\geq 1 + (1 + n)x$, as required.

(iv) We have that

$$2^{n+1} = 2 \cdot 2^n \geq 2(n + 1) = 2n + 2 = n + (n + 2) \geq n + 2.$$

(v) The number $n_0 = 4$. Then

$$(n + 1)! = (n + 1)n! \geq (n + 1)2^n \geq 2^n$$

since $n \geq 4$ and so certainly $n + 1 \geq 2$.

(vi) We have that

$$\sum_{i=1}^{n+1} 2i = \sum_{i=1}^{n} 2i + 2(n + 1)$$

which is equal to

$$n(n + 1) + 2(n + 1)$$

by (IH). This simplfies to

$$(n+1)(n+2)$$

as required.

(vii) We have that

$$\sum_{i=1}^{n+1}(2i)^2 = \sum_{i=1}^{n}(2i)^2 = (2(n+1))^2 = \frac{2}{3}n(n+1)(2n+1) + 4(n+1)^2$$

by (IH). We now just have to simplify the RHS. It is equal to

$$\frac{2}{3}(n+1)[n(2n+1) + 6(n+1)]$$

which in turn is equal to

$$\frac{2}{3}(n+1)(n+2)(2n+3)$$

as required.