Solutions: Exercises 4

1. (i) $v = 12$, $e = 23$, $f = 13$, and $f - e + v = 2$.

   (ii) $v = 6$, $e = 5$, $f = 1$, and $f - e + v = 2$.

   (iii) $v = 6$, $e = 5$, $f = 1$, and $f - e + v = 2$.

   (iv) $v = 6$, $e = 8$, $f = 4$, and $f - e + v = 2$.

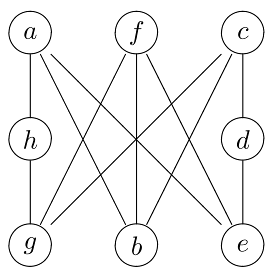
2. (i) The outside face has face-degree 10, the remaining 12 faces have face-degrees 3. Thus the sum of the face degrees is 46, which is twice the number of edges. This is because each edge is counted twice when the face-degrees are added up — either because the edge bounds two faces, or because it bounds only one face and is counted twice in the path around the face.

   (ii) The only face has face-degree 10 — twice the number of edges.

   (iii) Face-degree of 10.

   (iv) Outside face has degree 6, the remaining faces have degrees 4, 3 and 3. Their sum is 16 which is twice the number of edges.

3. Delete the edge hd and you have a graph which is a subdivision of $K_{3,3}$. Thus by Kuratowski's Theorem the original graph is not planar.



4. All trees are planar — it follows by Kuratowski's Theorem because if a tree contained a subdivision of either $K_{3,3}$ or K_5 then it would contain a cycle, which is impossible. However, it is possible to see that trees must be planar in a direct way. It can be done by drawing a tree as a genuinely tree-like graph. I omit the details.

5. K_1, K_2, K_3, K_4 are all planar as can easily be seen by redrawing them. We have proved that K_5 is not planar. For $n \geq 6$ none of the graphs K_n is planar because they contain K_5 as a subgraph.

6. The graphs $K_{m,n}$ where $1 \leq m, n \leq 2$ are all planar as can easily be seen by redrawing them. The graphs $K_{2,n}$ for all n are planar as can be seen by redrawing them. The graph $K_{3,2}$ is planar as can be seen by redrawing. We have proved that the graph $K_{3,3}$ is not planar. For $m, n \geq 3$ none of the graphs $K_{m,n}$ is planar because they contain $K_{3,3}$ as a subgraph.

7. Let the number of 5-cycles be x, the number of 6-cycles be y, and the number of faces be f. Thus $x+y=f$. Each vertex has degree 3. Suppose the number of vertices is v. Then by the Handshaking Lemma we have that $3v=2e$. The sum of all the face degrees is twice the number of edges and so $5x+6y=2e$. By Euler's formula $f-e+v=2$. We can eliminate v since $v=\frac{2}{3}e$. Thus Euler's forumla becomes $3f-e=6$. We therefore get the two equations
$$5x+6y=2e$$
and
$$3x+3y=6+e.$$
Using these two equations we get the remarkable result that $x=12$.

8. The proof runs as follows:
   - The graph is planar and so $f-e+v=2$.
   - We arrange this equation to get $e=f+v-2$.
   - Our goal is to eliminate f and get an inequality involving e and v.
   - By assumption, the length of the smallest cycle in the graph is g. It follows that each face-degree is at least g.
   - The sum of the face-degrees is $2e$.
   - There are f faces and each face has degree at least g, so that the sum of the face-degrees is at least fg. Hence $fg \leq 2e$.
   - It follows that $f \leq \frac{2e}{g}$; we can now eliminate f as we wanted.
   - It follows that $e \leq \frac{2e}{g}+v-2$.
   - Rearranging this inequality gives us the answer.