2008 Test 2/Solutions: F11ME3 Algebra 3/graph theory

1. 6.
   The point of this question is that to count the number of faces of a planar graph, you must first draw it without edges crossing. Alternatively, you can use Euler's formula.

2. 10.
   Calculating face degrees is usually straightforward, but where edges 'stick out' as in this graph, you must count both sides of the edge.

3. 40.
   Your spanning tree may differ from mine but the minimum weights will have to be the same. *Quite a number of you got this question wrong. This means you are not applying Prim's algorithm correctly.*

4. **Base case:** When $n = 1$, the LHS is equal to $4 - 6 = -2$, and the RHS is equal to $1 - 3 = -2$. Thus the base case holds.

   IH: Assume that $\sum_{i=1}^{n}(4 - 6i) = n - 3n^2$ for some n.

   Proof step: We prove that the formula holds for $n + 1$. We calculate

$$\begin{aligned}\sum_{i=1}^{n+1}(4-6i) &= \sum_{i=1}^{n}(4-6i) + (4 - 6(n+1)) \\ &= (n - 3n^2) + 4 - 6(n+1) \text{ by (IH)} \\ &= -3n^2 - 5n - 2 \\ &= (n+1) - 3(n+1)^2\end{aligned}$$

   Thus by the Principle of Induction we have proved that the formula always holds.

   You might find it helpful to use the following wording: in (IH): assume the formula holds for $n = k$; that is, $\sum_{i=1}^{k}(4-6i) = k-3k^2$. In the proof step, we prove that the formula holds for $n = k + 1$ that is we prove that $\sum_{i=1}^{k+1}(4 - 6i) = (k + 1) - 3(k + 1)^2$. Although not mathematically necessary, I think this phrasing has psychological benefits.