Axioms for high-school algebra

This is a version of Sections 4.1 and 4.2.

Key definitions

- Let X be a non-empty set. A *binary operation* $*$ on X associates with every ordered pair (x, y) of elements of X a single element $x * y$ of X.
- The binary operation $*$ is said to be *commutative* if $x * y = y * x$ for all $x, y \in X$.
- The binary operation $*$ is said to be *associative* if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.
- An *identity* for $*$ is an element e such that $e * x = x = x * e$ for all $x \in X$.
- Let e be an identity for $*$. Then the element x is said to be *invertible* if there is an element y such that $x * y = e = y * x$.

Algebraic properties of $\mathbb{R}$

We will only describe the properties of addition and multiplication. Subtraction and division are not viewed as binary operations in their own right. Instead, we *define*

$$a - b = a + (-b).$$

Thus to subtract b means the same thing as adding $-b$. Likewise, we *define*

$$a \backslash b = a \div b = a \times b^{-1} \text{ if } b \neq 0.$$

Thus to divide by b is to multiply by b^{-1}.

Axioms for addition in $\mathbb{R}$

(F1): *Addition is associative.* Let x, y and z be any numbers. Then $(x + y) + z = x + (y + z)$.

(F2): *There is an additive identity.* The number 0 (zero) is the additive identity. This means that $x + 0 = x = 0 + x$ for any number x.

(F3): *Each number has a unique additive inverse.* For each number x there is a number, denoted $-x$, with the property that $x + (-x) = 0 = (-x) + x$. The number $-x$ is called the *additive inverse* of the number x.

(F4): *Addition is commutative.* Let x and y be any numbers. Then $x + y = y + x$.

Axioms for multiplication in $\mathbb{R}$

(F5): *Multiplication is associative.* Let x, y and z be any numbers. Then $(xy)z = x(yz)$.

(F6): *There is a multiplicative identity.* The number 1 (one) is the multiplicative identity. This means that $1x = x = x1$ for any number x.

(F7): *Each non-zero number has a unique multiplicative inverse.* For each non-zero number x there is a unique number, denoted x^{-1}, with the property that $x^{-1}x = 1 = xx^{-1}$. The number x^{-1} is called the *multiplicative inverse* of x. It is, of course, the number $\frac{1}{x}$, the *reciprocal* of x.

(F8): *Multiplication is commutative.* Let x and y be any numbers. Then $xy = yx$.

Linking axioms for $\mathbb{R}$

(F9): $0 \neq 1$.

(F10): *The additive identity is a multiplicative zero.* This means that $0x = 0 = x0$ for any x.

(F11): *Multiplication distributes over addition on the left and the right.* There are two distributive laws: the *left distributive law* $x(y + z) = xy + xz$ and the *right distributive law* $(y + z)x = yx + zx$.

In fact, many other structures satisfy all these axioms. Anything that does is called a *field*.

We have missed out one further ingredient in algebra, and that is the properties of equality. The following are not exhaustive but enough for our purposes.

Axioms for equality

(E1): $a = a$.

(E2): If $a = b$ then $b = a$.

(E3): If $a = b$ and $b = c$ then $a = c$.

(E4): If $a = b$ and $c = d$ then $a + c = b + d$.

(E5): If $a = b$ and $c = d$ then $ac = bd$.