F17CC *Each question part is worth 5 marks.*

1. (a) A set X contains 15 elements. How many subsets does it have?

   (b) Write the complex number $\frac{1}{6+4i}$ in the form $a+bi$ where a and b are real numbers.

   (c) Carry out the following matrix multiplication

$$\begin{pmatrix} 1 & 4 & -1 \\ 2 & 0 & 3 \\ -5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & 3 \\ 3 & 4 & 1 \end{pmatrix}$$

   (d) Find the angle to the nearest degree between the vectors **a** and **b** where

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

2. (a) How many committees of 5 people can be formed from 120 bureaucrats?

   (b) Find the square roots of $-16 - 30i$ and show that your solutions work.

   (c) Solve the following system of linear equations. You **must** use elementary row operations. Show that your solutions work.

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ -3x + 2y - 2z &= -10. \end{aligned}$$

   (d) Calculate the vector product $\mathbf{a} \times \mathbf{b}$ of the vectors **a** and **b** where

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Exam paper continues ...

3. (a) Factorize $x^4 + x^2 + 1$ as a product of two real irreducible quadratic polynomials.

   (b) Find the 4th roots of 3.

   (c) Find the inverse of the matrix below **by first finding its adjugate**, and show that your answer works

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 2 \end{pmatrix}.$$

   (d) Find the Cartesian equation of the plane that passes through the point with position vector $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and has $\mathbf{d} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ as its norma1.

4. (a) Show that the set of solutions to the following system of linear equations

$$\begin{aligned} 2x + y - 3z &= 0 \\ 4x + 2y - 6z &= 0 \\ x - y + z &= 0 \end{aligned}$$

   forms a line through the origin.

   (b) Prove by induction that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

   (c) Prove that for complex numbers u and v we have that $|u||v| = |uv|$.

   (d) Prove Pythagoras' theorem using vectors.

End of paper

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section and page numbers refer to my book

1. (a) 2^{15}. [page 75]
   (b) $\frac{3}{26} - \frac{1}{13}i$. [page 168]
   (c)
   $$\begin{pmatrix} -1 & 9 & 9 \\ 13 & 14 & -1 \\ -4 & 21 & 30 \end{pmatrix}$$
   [pages 221–224]
   (d) $73°$. [page 228, page 297, Question 1(e) of Exercises 9.2]

2. (a) $\binom{120}{5}$. [page 77]
   (b) $\pm(3-5i)$. One mark for showing that solutions work. [page 170]
   (c) $x = 2$, $y = -1$, $z = 1$. One mark for showing that solutions work. [Section 8.3]
   (d) $5\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$. [pages 291–292, page 297]

3. (a) $(x^2-x+1)(x^2+x+1)$. This question requires a little thought and there are a number of ways of solving it. A clever way is to notice that $x^4+x^2+1 = (x^2+1)^2-x^2$. Alternatively, you could put $y = x^2$ and find both roots of y^2+y+1 and then find the square roots of the two answers. Finally, write $x^4+x^2+1 = (x^2+ax+b)(x^2+cx+d)$ and solve for a, b, c, d. [Section 4.3, page 186]
   (b) The 4th roots of unity are ± 1, $\pm i$. The principal 4th root of 3 is $\sqrt[4]{3}$. Thus the 4th roots of 3 are: $\pm\sqrt[4]{3}$, $\pm\sqrt[4]{3}i$. [Section 7.5]
   (c) The inverse of the matrix is
   $$\begin{pmatrix} 2 & 0 & -1 \\ 3 & -1 & -1 \\ -2 & 1 & 1 \end{pmatrix}$$
   One mark for showing that solution works. [pages 253–260]
   (d) $x + 2y + 3z = 6$. [pages 307–308]

4. (a) The solution set is the line through the origin with parametric equation $\lambda(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$. [Section 8.3]

(b) Base case: formula works when $n = 1$. The induction step amounts to showing that $(n+1)^3 + \frac{1}{4}n^2(n+1)^2 = \frac{1}{4}(n+1)^2(n+2)^2$. [Section 3.8]

(c) Let $u = a+bi$ and $v = c+di$. Then the proof amounts to showing that $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$. [Lemma 6.1.6]

(d) Let the legs of the perpendicular be $\mathbf{a}$ and $\mathbf{b}$ and let the longest side be $\mathbf{c}$ where $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$. Therefore $\mathbf{a}+\mathbf{b} = -\mathbf{c}$. Take inner products of both sides to get $\mathbf{a}^2 + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2 = \mathbf{c}^2$. But $\mathbf{a}\cdot\mathbf{b} = 0$, since triangle is right-angled. Now observe that $\mathbf{x}^2$ is just the length of $\mathbf{x}$ squared. [Example 9.1.7]