HERIOT WATT UNIVERSITY

SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

F17CC

Introduction to University Mathematics

Semester 1 – 2019/20

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

Each question part is worth 5 marks.

1. (a) How many possible outcomes are there when a coin is tossed 15 times [3 marks]? Of these, how many outcomes contain exactly 5 heads [2 marks]?

   (b) Write the complex number $\frac{1+7i}{2+10i}$ in the form $a+bi$ where a and b are real numbers.

   (c) Let
$$A = \begin{pmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{pmatrix}.$$
   Calculate $A^3 - 8A^2 + 5A$.

   (d) Find the angle to the nearest degree between the vectors $\mathbf{a}$ and $\mathbf{b}$ where
$$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ and } \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}.$$

2. (a) What is the constant term of
$$\left(2x + \frac{1}{x^2}\right)^{12}?$$

   (b) Find the square roots of $56 + 90i$ [4 marks] and show that your solutions work [1 mark].

   (c) Solve the following system of linear equations [4 marks]. You **must** use elementary row operations. Show that your solutions work [1 mark].
$$\begin{aligned} -x + y + 2z &= 2 \\ 3x - y + z &= 6 \\ -x + 3y + 4z &= 4. \end{aligned}$$

   (d) Calculate the vector product $\mathbf{a} \times \mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$ where
$$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ and } \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}.$$

Exam paper continues . . .

3. (a) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. List the elements of the set
$$(A \setminus B) \times (B \setminus A).$$
(b) Factorize $x^4 - 4x^3 + 7x^2 - 8x + 4$ as a product of real linear and real irreducible quadratic polynomials [4 marks]. You should show explicitly that any real quadratics are irreducible [1 mark].
(c) By expanding $(\cos\theta + i\sin\theta)^6$ in two different ways, obtain an expression for $\cos 6\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.
(d) Calculate the determinant of the following matrix
$$\begin{pmatrix} 7 & 14 & 21 \\ 36 & 18 & 6 \\ 87 & 12 & -45 \end{pmatrix}.$$

4. (a) Find all solutions to the following system of linear equations [4 marks]. You **must** use elementary row operations. Show that your solutions work [1 mark].
$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + z &= 3 \\ 3x + 3y + 2z &= 4. \end{aligned}$$
(b) Find the inverse of the following matrix
$$\begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix}.$$
(c) Find the *non-parametric equation* of the plane that contains the points given by the following three position vectors:
$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$
(d) Find the characteristic polynomial and eigenvalues (including their multiplicities) of the following matrix
$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}.$$

End of paper

SOLUTIONS TO EXAM PAPER 2019: all questions apply standard methods. For details see lecture notes/my book.

1. (a) 2^{15}. $\binom{15}{5}$.
   (b) $\frac{9}{13} + \frac{1}{26}i$.
   (c) $-7I$.
   (d) $85°$.

2. (a) This is an application of the binomial theorem. The constant term is $2^8\binom{12}{4}$ which is equal to $126,720$.
   (b) $\pm(9+5i)$. One mark for showing that solutions work.
   (c) $x = 1$, $y = -1$, $z = 2$. One mark for showing that solutions work.
   (d) $5\mathbf{i} + 16\mathbf{j} + 17\mathbf{k}$.

3. (a) $\{1, 2\} \times \{5, 6\}$.
   (b) $(x-1)(x-2)(x^2 - x + 2)$. The discriminant of the real quadratic is $(-1)^2 - 4 \cdot 2 < 0$ and so is irreducible.
   (c) By De Moivre's theorem we get $\cos 6\theta + i \sin 6\theta$. By the Binomial theorem we get $\sum_{j=0}^{6} \binom{6}{j} \cos^{6-j}\theta (i \sin\theta)^j$. Now equate real values on each side to obtain

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta.$$

   (d) 0.

4. (a) The solutions are

$$\begin{pmatrix} 2-\lambda \\ \lambda \\ -1 \end{pmatrix}$$

   where $\lambda \in \mathbb{R}$.
   (b) The inverse is

$$\begin{pmatrix} -\frac{7}{10} & \frac{1}{5} & \frac{3}{10} \\ -\frac{13}{10} & -\frac{1}{5} & \frac{7}{10} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}.$$

(c) $-x+2y-z=0$. [You can check that the three original points satisfy this equation].

(d) The characteristic polynomial is $-(x-3)^2(x-5)$ and so the eigenvalues are 3 (twice) and 5.