F17CC IUM Homework 1: Solutions

1. $$(4x-6y)^{42} = (4x+(-6y))^{42}$$
$$= \sum_{i=0}^{42} \binom{42}{i} (4x)^i (-6y)^{42-i}$$

by the binomial theorem (not Pascal's $\triangle$)

$$= \sum_{i=0}^{42} \underbrace{\binom{42}{i} 4^i (-6)^{42-i}}_{\text{coefficient} = \text{number}} x^i y^{42-i}$$

Put $i=30$. Required coefficient is

$$\binom{42}{30} 4^{30} 6^{12}$$

You should not attempt to calculate explicitly what this number is (completely pointless) nor should you express it as a power of 10 (which is only approximate).

2. We need to find the (two) complex numbers $x+yi$ such that $(x+yi)^2 = -56+390i$. $(*)$.

[You cannot use De Moivre's theorem to get an exact answer to this question.]

Expand LHS of $(*)$ and equate real and imaginary parts to get the following two equations

(1) $x^2 - y^2 = -56$.

(2) $2xy = 390$.

(3) This equation is obtained by taking moduli of both sides of $(*)$

$$x^2+y^2 = |-56+390i|$$
$$= 394$$

(1) + (3) gives $x^2 = 169$ and so $x = \pm 13$.

By (2), $y = \frac{195}{x}$.

When $x = 13$ then $y = 15$,

When $x = -13$ then $y = -15$.

Thus the roots are $\pm(13+15i)$

Now check solutions by showing that

$$(13+15i)^2 = -56+390i.$$

3. By de Moivre's theorem

$$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta.$$

Now expand the LHS above using the binomial theorem to get

$$(\cos\theta + i\sin\theta)^6 = \sum_{j=0}^{6} \binom{6}{j} \cos^j\theta \, (i\sin\theta)^{6-j}$$

I changed variable to avoid notation clash

It follows that

$$\cos 6\theta + i\sin 6\theta = \sum_{j=0}^{6} \binom{6}{j} \cos^j\theta \quad (i\sin\theta)^{6-j}$$

Now equate real parts of each side to get (after a little work)

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta.$$

4. We begin by finding the integer roots of

$$x^5 - 13x^4 + 60x^3 - 130x^2 + 144x - 72. \quad (*).$$

We do this by testing the divisors of -72.

We find that $2, 3, 6$ are all roots.

Divide $(*)$ by $(x-2)(x-3)(x-6)$ (either one by one or as a product). What's left is the quotient $x^2 - 2x + 2$. Find the roots of this quadratic in the usual way. It follows that the roots of $(*)$ are $2, 3, 6, 1+i, 1-i$.

Now, $x^2 - 2x + 2$ has discriminant < 0. Thus it is a real irreducible quadratic. It follows that the desired factorization of $(*)$ is

$$(x-2)(x-3)(x-6)(x^2-2x+2).$$