Lecture 2

1. Combinatorics

This subject is about counting. It is therefore particularly important in probability theory

Goal: Develop formulae for counting permutations and combinations

This will require: Counting principles (to help to count).

Therefore we need: set notation to formulate these counting principles precisely.

We therefore begin with set notation

Aside Hallmarks (i.e. distinguishing features of university Maths:

- Precision. We always need to be clear and unambiguous. To help us, we develop notation systems and formulate definitions.

- Proofs. How do you know what you are doing is correct? A proof shows that a formula works or a method does what is advertized. Any rational entity can check proofs for themselves. [Maths is not about opinions].

Basic set notation

See Section 3.1 of my book (you have access to a PDF file) for more information.

A *Set* is a collection of things, called its *elements*, which we wish to regard as a whole.

Two sets are *equal* when they contain the same elements.

<u>Example</u> Here are some random things:

$\bigcirc$ $\uparrow$

$\square$ $\ast$

For the purposes of this lecture, I want to be able to talk about these things (i.e. regard them as a single "thing").

I therefore gather them together into a set

Commas separate the elements of this set

I can give this set <u>a name</u>

$$\{\square, \bigcirc, \ast, \uparrow\} = A$$

braces/curly brackets delimit the set

I can now talk about the set A and you know what I mean (because I have told you what A is).

The <u>order</u> in which you list the elements of the set does not matter. So,

$$A = \{\uparrow, *, \circ, \square\}$$

even though the elements are listed in a different order. [A set is a bag not a display case.]

The things that make up the set are called its <u>elements</u>.

We write

$\uparrow \in A$ ('$\uparrow$ <u>is an element of</u> A')

shorthand

We write

$$\underline{\notin}\ A \quad (\text{'}\Delta \text{ \underline{is not an element of} } A\text{'})$$

An important feature of set notation that seems odd when first encountered is that <u>repetitions of elements are ignored</u>

<u>Example</u> $\{1,1\} = \{1\}$.

The set $\{\}$ has nowt in it.
It is called the <u>empty set</u>. To avoid any misunderstanding, we give this set a name: $\emptyset$

<u>Example</u> $\emptyset \neq \{\emptyset\}$

LHS is a set with no elements.
RHS is a set with exactly one element (namely $\emptyset$).

Set notation seems simple, even simple-minded. But we can form sets whose elements are themselves sets.

<u>Example</u> $X = \{\emptyset, \{a\}, \{\{a\}\}, \{a,b\}\}$

is a set with four elements.

$$\boxed{\text{All of Maths can be described using sets.}}$$

<u>Definition</u> Let A be a set. The <u>cardinality</u> of A, written $|A|$, is simply the number of elements in A.

<u>Examples</u>

(i) $|\emptyset| = 0$.

(ii) $|\{1,2,3\}| = 3$.

(iii) $|\{a,b\}| = 2$.

<u>Remark</u> Probability theory often involves calculating $|A|$ for special sets A. (I'll give some examples later).

Some special sets with special names

$\mathbb{N} = \{0, 1, 2, \underbrace{\ldots}_{\text{read 'and so on'}}\}$ the natural numbers

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ the integers (Zahlen – German for numbers)

$\mathbb{Q}$ all positive and negative fractions or quotients. The rational numbers.

$\mathbb{R}$ all real numbers (can be written $\underset{\substack{\uparrow \\ \text{integer}}}{a} \cdot \underbrace{a_1 a_2 a_3 \ldots}_{\text{potentially infinite}}$)

A very common way to define a set is to ~~describe~~ use a property that defines the elements of the set.

Example

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, \dots\}$$

$$= \{a \underset{\text{such that}}{:} \underbrace{a \text{ is a prime}}_{\text{property a has to have}}\}$$

Examples

(i) $A = \{a : a \in \mathbb{R} \text{ and } a^2 = 1\} = \{-1, 1\}$

(ii) $B = \{a : a \in \mathbb{N} \text{ and } a^2 = 1\} = \{1\}$

(iii) $C = \{a : a \in \mathbb{Q} \text{ and } a^2 = 2\} = \emptyset$ ← proof needed

(iv) $D = \{a : a \in \mathbb{R} \text{ and } a^2 = -1\} = \emptyset$.

Let A be a set.
We can form sets B whose elements are only chosen from A. We say that B is a <u>subset of</u> A, written $A \subseteq B$

<u>Examples</u>

(1) $\emptyset \subseteq A$. <u>Choose</u> no elements from A

(2) $A \subseteq A$ Choose all elements from A.

<u>Definition</u> Let A be a set.

$P(A) = \{B : B \subseteq A\}$ is called the <u>power set</u> of A; it is the set whose elements are all of the subsets of A.

Example Let $A = \{1, 2, 3\}$.

Find $P(A)$. [Be systematic]

Cardinality of subset	subsets
0	$\emptyset$
1	$\{1\}, \{2\}, \{3\}$
2	$\{1,2\}, \{1,3\}, \{2,3\}$
3	$\{1,2,3\}$

$\therefore$

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Observe that

$$\left| P(\{1,2,3\}) \right| = 8 = 2^3.$$

There is an interesting pattern in the number of subsets of a given size.

Cardinality of subsets	Number of such subsets
0	1
1	3
2	3
3	1

So, $8 = 1 + 3 + 3 + 1$

Pascal's Δ

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \leftarrow !
\end{array}
$$

<u>This is not a coincidence.</u>

$\boxed{\text{Why does the power set "know" about Pascal's } \Delta \text{?}}$