

Lecture 2

1. Combinatorics

This subject is about Counting. It is therefore particularly important in Probability theory

Goal : Develop formulae for counting permutations and combinations

This will require: Counting principles (to help to count).

Therefore we need: Set notation to formulate these counting principles precisely.

We therefore begin with set notation

Aside Hallmarks (i.e. distinguishing features) of university Maths:

- Precision. We always need

to be clear and unambiguous.

To help us, we develop notation systems and formulate definitions.

- Proofs. How do you know what you are doing is correct? A proof shows that a formula works or a method does what is advertised. Any rational entity can check proofs for themselves.
[Maths is not about opinions].

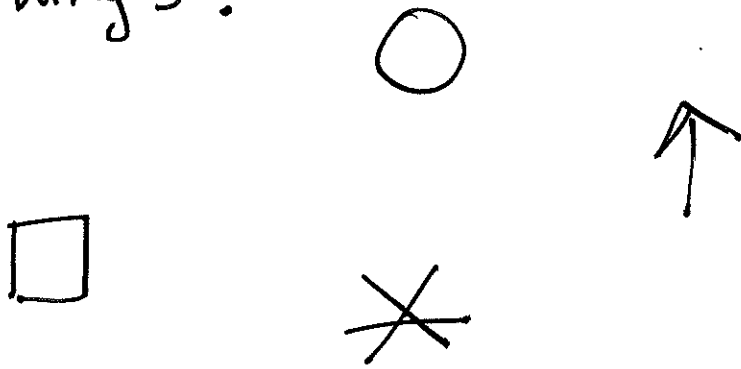
Basic Set notation

See Section 3.1 of my book
(you have access to a PDF file) for
more information.

A Set is a collection of things,
called its elements, which we wish
to regard as a whole.

Two sets are equal when they contain
the same elements.

Example Here are some random things:



For the purposes of this lecture, I want to be able to talk about these things (i.e. regard them as a single "thing").

I therefore gather them together ~~into~~ into a set
 Commas separate the elements of this set

$$\{ \square, \circ, *, \uparrow \} = A$$

I can give this set a name

braces/curly brackets delimit the set

We write

$$\Delta \notin A \quad / \quad \Delta \text{ is not an element of } A)$$

An important feature of set notation that seems odd when first encountered is that repetitions of elements are ignored

Example $\{1, 1\} = \{1\}$.

The set $\{\}$ has no elements.

It is called the empty set. To avoid any misunderstanding, we give this set a

name: $\emptyset$

Example $\emptyset \neq \{\emptyset\}$

LHS is a set with no elements.

RHS is a set with exactly one element (namely $\emptyset$).

Set notation seems simple, even simple-minded.

But we can form sets whose elements are themselves sets.

Example $X = \{\emptyset, \{a\}, \{\{a\}\}, \{a, b\}\}$

is a set with four elements.

All of Maths can be described using sets.

Definition Let A be a set. The cardinality of A , written $|A|$, is simply the number of elements in A .

Examples

$$(i) \quad |\emptyset| = 0.$$

$$(ii) \quad |\{1, 2, 3\}| = 3.$$

$$(iii) \quad |\{a, b\}| = 2.$$

Remark Probability theory often involves calculating $|A|$ for special sets A .
(I'll give some examples later).

Some special sets
with special names

$\mathbb{N} = \{0, 1, 2, \dots\}$ the natural numbers
 read 'ad soon'

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

the integers (Zahlen - German for numbers)

~~$\mathbb{Q}$~~ all positive and negative fractions or quotients. The rational numbers.

$\mathbb{R}$ all real numbers (can be written
 $\$ a = a_1 a_2 a_3 \dots$
 $\uparrow$ integer part (potentially infinite))

A very common way to define a set is to ~~define~~ use a property that defines the elements of the set.

Example

$$P = \{2, 3, 5, 7, 11, 13, \dots\}$$

$$= \{a : a \text{ is a prime}\}$$

such that

property a has to have

Examples

$$(i) A = \{a : a \in \mathbb{R} \text{ and } a^2 = 1\} = \{-1, 1\}$$

$$(ii) B = \{a : a \in \mathbb{N} \text{ and } a^2 = 1\} = \{1\}$$

$$(iii) C = \{a : a \in \mathbb{Q} \text{ and } a^2 = 2\} = \emptyset \leftarrow \text{proof needed}$$

$$(iv) D = \{a : a \in \mathbb{R} \text{ and } a^2 = -1\} = \emptyset$$

Let A be a set.

We can form sets B whose elements are only chosen from A . We say that B is a subset of A , written $A \subseteq B$.

Examples

(1) $\emptyset \subseteq A$. choose no elements from A

(2) $A \subseteq A$ Choose all elements from A .

Definition Let A be a set.

$P(A) = \{B : B \subseteq A\}$ is called

the power set of A ; it is the set whose elements are all of the subsets of A .

Example

Let $A = \{1, 2, 3\}$.

Find $P(A)$. [Be systematic]

Cardinality of subset	subsets
0	$\emptyset$
1	$\{1\}, \{2\}, \{3\}$
2	$\{1, 2\}, \{1, 3\}, \{2, 3\}$
3	$\{1, 2, 3\}$

$$P(A) = \left\{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \right\}$$

Observe that

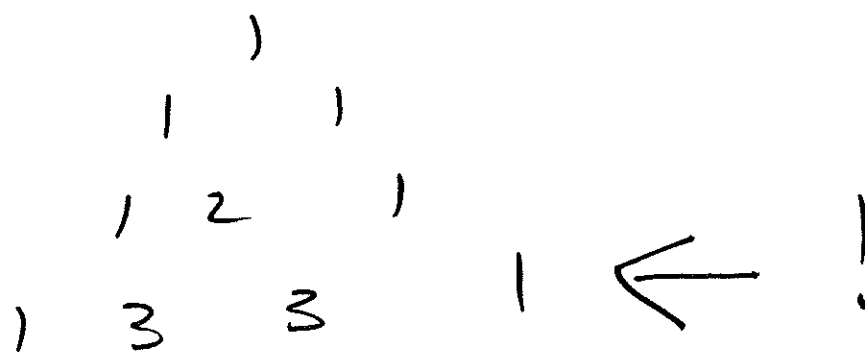
$$|P(\{1, 2, 3\})| = 8 = 2^{\underline{3}}$$

There is an interesting pattern in the number of subsets of a given size.

Cardinality of subsets	Number of such subsets
0	1
1	3
2	3
3	1

$$\text{So, } 8 = 1 + 3 + 3 + 1$$

Pascal's Δ



This is not a coincidence.

Why does the power set
"know" about Pascal's Δ ?