The characteristic polynomial

This section is just to give a flavour — maybe, a smell — of matrix theory.

Definition Let A be a square matrix.

Define

$$\chi_A(x) = \det(A - xI)$$

is a polynomial called the characteristic polynomial of A. [χ is the Greek letter chi].

The roots of $\chi_A(x)$ are called the eigenvalues of A. These contain important information about A useful in many applications.

<u>Example</u> Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Then
$$\chi_A(x) = \det(A - xI)$$
$$= \det \begin{pmatrix} 1-x & 1 & 1 \\ 0 & 1-x & 1 \\ 0 & 0 & 1-x \end{pmatrix}$$
$$= \underline{\underline{(1-x)^3}}$$

Eigenvalues = 1 (thrice).

<u>Theorem</u> (Cayley-Hamilton).

Each square matrix is a <u>root</u> of its characteristic polynomial. [The word <u>root</u> is used here in an extended sense].

<u>Example</u> Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Its characteristic polynomial is

$$\chi_A(x) = \begin{vmatrix} 1-x & 2 \\ 2 & 3-x \end{vmatrix} = x^2 - 4x \cancel{+} - 1.$$

To understand what $\chi_A(A)$ means, we ~~must~~ need to remember that

$$x^2 - 4x - 1 = x^2 - 4x^1 - x^0.$$

Define $A^0 = I$, the identity matrix the same size of A

Thus $\chi_A(A) = A^2 - 4A - I$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, A is a root of $\chi_A(x)$. $\blacksquare$

END

OF

SECTION 3