Lecture 10

Complex numbers

Let's go back to the formula for the roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (\text{where } a, b, c \in \mathbb{R},\ a \neq 0).$$

These are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (*)$$

Recall that $D = b^2 - 4ac$ is called the discriminant.
We are now interested in the irreducible quadratic when $D < 0$. In this case, we can write

$$\boxed{x = \frac{-b}{2a} \pm \frac{\sqrt{-|D|}}{2a}} \qquad \left\{ \begin{array}{l} |D| > 0 \\ \text{so } -|D| < 0. \end{array} \right.$$

$$\sqrt{-|D|} = \sqrt{-1}\ \sqrt{|D|}$$

we ~~set~~ put $\boxed{i = \sqrt{-1}}$ $\boxed{i^2 = -1}$

So, $$x = \frac{-b}{2a} \pm \frac{i\sqrt{|D|}}{2a}.$$

A "number" of the form $x+iy$ (or $x+yi$) where $x,y \in \mathbb{R}$ is called a <u>Complex number</u>.

The set of complex numbers is denoted by $\mathbb{C}$.

$$\mathbb{C} = \{a+ib : a,b \in \mathbb{R}\}.$$

Observe that $\mathbb{R} \subseteq \mathbb{C}$ since $r = r + 0i$.

<u>Remark</u> This is less mysterious than it looks. When we write $\sqrt{2}$ <u>we mean</u> the real number with the property that $(\sqrt{2})^2 = 2$. When we write i <u>we mean</u> that symbol with the property that $i^2 = -1$.

Observe that the square roots of -1 are then $\pm i$.

<u>Example</u> Calculate the complex roots of $x^2 - 2x + 2$ (*). Using the usual formula for the roots of a quadratic we get that

$$x = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= 1 \pm \tfrac{1}{2}\sqrt{-4}$$

$$= \underline{\underline{1 \pm i}}$$

Check that $1-i$ is also a root

We now check that these roots work.

We show that $1+i$ is a root of (*)

$$(1+i)^2 - 2(1+i) + 2$$

$$= \cancel{1} + \cancel{2i} + \cancel{i^2}^{-1} \cancel{-2} \cancel{-2i} \cancel{+2} = 0$$

Using complex numbers all real numbers have two roots (except 0 which has one).

<u>Example</u> The square roots of -3 are $\pm i\sqrt{3}$. Since,

$$(i\sqrt{3})^2 = i^2(\sqrt{3})^2 = -1 \cdot 3 = -3$$

If $a+ib$ and $c+id$ are complex numbers then $a+ib = c+id$ iff $a=c$ and $b=d$.

It turns out that $\mathbb{C}$ is a <u>field</u> (i.e. satisfies the same algebraic properties as $\mathbb{R}$).

Addition

$$(a+ib)+(c+id) = a+ib+c+id$$

$$= a+c+ib+id = (a+c)+i(b+d)$$

Multiplication

$$(a+ib)(c+id) = (a+ib)c+(a+ib)id$$

$$= ac+ibc+aid+ibid$$

$$=(ac-bd)+i(bc+ad).$$

Division (except by zero).

To define this, we need an auxiliary concept.

Let $z = a + ib$.

Define $\overline{z} = a - ib$.

This operation is called the <u>Complex Conjugate</u> of z. Now observe why it is important

$$z\overline{z} = (a+ib)(a-ib)$$
$$= (a+ib)a - (a+ib)ib$$
$$= a^2 + \cancel{iba} - \cancel{abi} + b^2$$
$$= a^2 + b^2.$$

It follows that $z\overline{z} \in \mathbb{R}$ and $z\overline{z} \geqslant 0$.

$(z\overline{z} = 0 \iff a = 0 \text{ and } b = 0 \iff z = 0)$.

Now let $a+ib \neq 0$ (ie $a \neq 0$ & $b \neq 0$).

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)}.$$

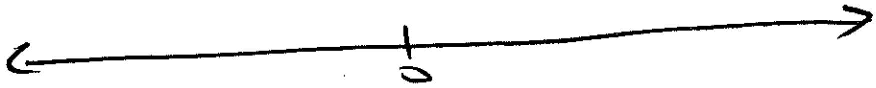
$$= \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - \frac{b\,i}{a^2+b^2}$$

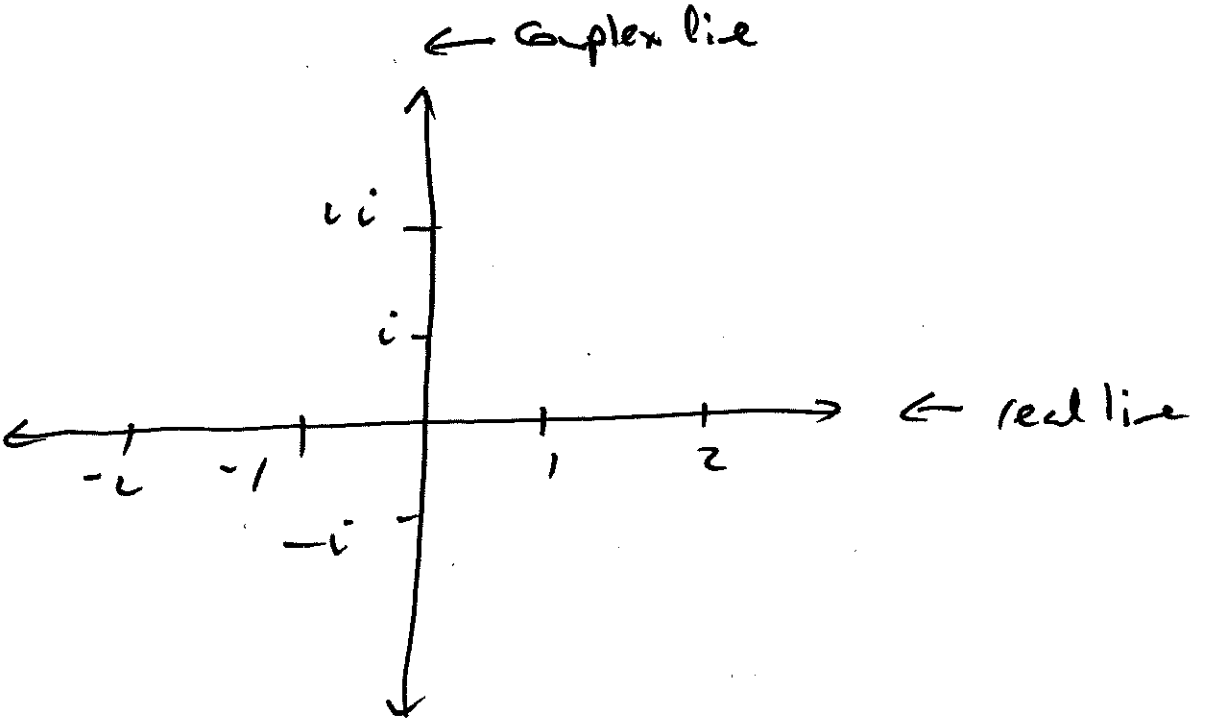
~~So, The divide by a+ib must~~

We can now carry out all four operations / addition subtraction multiplication, and division, and these operations have the usual properties (they form a <u>field</u>).

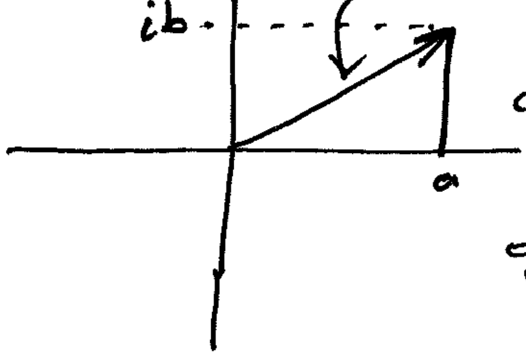
Where do complex numbers live? The real numbers live on the real line



The complex numbers live in the complex plane



The complex number $a + bi$ represents a point in the complex plane — or, better, the line from the origin to that point



length $= \sqrt{a^2+b^2}$
$= |z| = \sqrt{z\bar{z}}$
called the Modulus of z.