lecture 11

Definition Let $z = a+ib$ be a complex number.

Define $|z| = \sqrt{a^2+b^2}$ — a real number —

called the modulus of z.

$\leftarrow$ helping result

$$\boxed{\textbf{Lemma} \quad |uv| = |u| \cdot |v|.}$$

Proof Let $u = a+ib$ and $v = c+id$.

Then $uv = (ac-bd) + (ad+bc)i$

- $|uv| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$

$$= \sqrt{(ac)^2 - \cancel{2(ac)(bd)} + (bd)^2 + (ad)^2 + \cancel{2(ad)(bc)} + (bc)^2}$$

$$= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$$

- $|u|\,|v| = \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}$

$$= \sqrt{(ac)^2 + (ad)^2 + (bc)^2 + (bd)^2} \quad \blacksquare$$

<u>Theorem</u> Every non-zero complex number $a+ib$ has exactly two roots.

<u>Proof</u> Let $(x+iy)^2 = a+ib$. $(*)$

Then $(x+iy)^2 = x^2 + 2xyi - y^2$

$\therefore \quad x^2 - y^2 + 2xyi = a+ib.$

Compare real and imaginary parts:

① $x^2 - y^2 = a$

② $2xy = b$.

We shall now get a third equation:

By $(*)$ $\quad |(x+iy)^2| = |a+ib|$

$\therefore$ by Lemma, $\quad |x+iy|^2 = |a+ib|$

So

③ $x^2 + y^2 = \sqrt{a^2+b^2}$

① $x^2 - y^2 = a$,

② $2xy = b$.

③ $x^2 + y^2 = \sqrt{a^2+b^2}$

① + ③ $2x^2 = a + \sqrt{a^2+b^2}$

$\therefore \quad x^2 = \frac{1}{2}\left(a + \sqrt{a^2+b^2}\right)$

& $x = \pm\sqrt{\frac{1}{2}\left(a + \sqrt{a^2+b^2}\right)}$

Now use ② to get $y = \frac{b}{2x}$

Two values of x lead to corresponding two values of y $\boxtimes$

Example Find the square roots of $3+4i$

$(x+iy)^2 = 3+4i$.

$x^2-y^2+2xyi = 3+4i$

① $x^2-y^2 = 3$

② $2xy = 4$

③ $x^2+y^2 = \sqrt{9+16} = 5$

① + ③ $2x^2 = 8$, $x^2 = 4$ $\underline{x = \pm 2}$

$y = \frac{4}{2x} = \frac{2}{x} = \pm 1$

$\therefore$ $\underline{x+iy = \pm(2+i)}$

Check $(2+i)^2 = 4+4i-1 = \underline{\underline{3+4i}}$