Lecture 12 Complex numbers continued...

Solve the following quadratic equation

$$x^2 + 4ix + (4i-1) = 0.$$

By the usual formula (works because $\mathbb{C}$ like $\mathbb{R}$ is a field)

$$x = \frac{-4i \pm \sqrt{(4i)^2 - 4(4i-1)}}{2}$$

$$= \frac{-4i \pm \sqrt{-12-16i}}{2}$$

$$= \frac{-4i \pm \sqrt{-4i}\sqrt{3+4i}}{2}$$

$$= \frac{-4i \pm 2i\sqrt{3+4i}}{2}$$

$$= -2i \pm i\sqrt{3+4i}$$

$$= -2i \pm i(2+i)$$ ← by a previous calculation

$$= -2i \pm (2i-1)$$

$$= \underline{\underline{-1 \text{ or } -4i+1}}$$

Then check that both of these numbers really are roots of the original quadratic. ▨

$\underline{\underline{-1}}$: $\cancel{1} - \cancel{4i} + \cancel{4i} \cancel{-1} = 0$

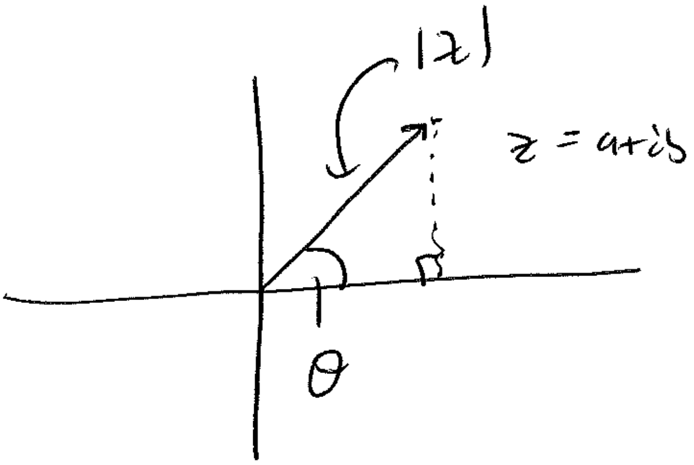
$-4i+1$ //:

$$(-4i+1)^2 + 4i(-4i+1) + (4i-1)$$

$$= (\cancel{16i^2} - \cancel{8i} + \cancel{1}) - \cancel{16i^2} + \cancel{4i} + \cancel{4i} \cancel{-1}$$

$$= 0$$

The geometry of complex numbers



Let $z \neq 0$.

Then $z = |z| (\cos \theta + i \sin \theta)$

by basic trig. This is called the polar form of the complex number z. The angle θ, or more accurately the angles $\theta + 2\pi n$ where $n \in \mathbb{Z}$, is called / are called the argument / arguments of z.

The polar form is therefore <u>not</u> unique (unlike the form $a + ib$). Compare fractions: the same fraction can have different names — $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$ etc.

This geometric way of thinking about complex numbers gives their theory a "different feel" from that of the real numbers.

Multiplication of complex numbers has a natural interpretation when viewed geometrically.

Proposition Let
$$u = |u| (\cos\theta + i\sin\theta)$$
and
$$v = |v| (\cos\phi + i\sin\phi).$$
Then $uv = |u||v| \left(\cos(\theta+\phi) + i\sin(\theta+\phi) \right)$

i.e. multiply the moduli and add the arguments.

<u>Proof</u>

$$
\begin{aligned}
uv &= |u| (\cos\theta + i\sin\theta)\, |v| (\cos\phi + i\sin\phi) \\
&= |u|\,|v| \left[(\cos\theta\cos\phi - \sin\theta\sin\phi) \right. \\
&\qquad \left. + i(\cos\theta\sin\phi + \sin\theta\cos\phi) \right] \\
&= |u|\,|v| \left[\cos(\theta+\varphi) + i\sin(\theta+\varphi) \right]
\end{aligned}
$$

by trig identities ■

__Example__ $i \times z$ rotates z 90° in an anticlockwise direction.

Thus $i^2 \times z$ rotates z 180° in an anticlockwise direction which is equal to $-1 \times z$. This gives us a way to picture the sense in which $i^2 = -1$.

We make the following deductions from our ~~proposition~~ proposition above.

Let $u = |u| \, (\cos\theta + i\sin\theta)$

Then $u^2 = |u|^2 \, (\cos 2\theta + i\sin 2\theta)$

$u^3 = |u|^3 \, (\cos 3\theta + i\sin 3\theta)$.

More generally,

$$u^n = |u|^n \, (\cos n\theta + i\sin n\theta)$$

A special case of the above result is

<u>De Moivre's Theorem</u>

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$n \geq 0$, $n \in \mathbb{N}$.

Complex numbers "know" the trig identities.

<u>Example</u> By De Moivre

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta \quad (*)$$

Binomial theorem

But $(\cos\theta + i\sin\theta)^2 =$

$$\cos^2\theta + 2\cos\theta\sin\theta\, i - \sin^2\theta$$

$$= (\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta\, i$$

Now compare real and imaginary parts

$$\boxed{\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\cos\theta\sin\theta\end{aligned}}$$

Using the Binomial theorem this can clearly be generalized to the case $(\cos\theta + i\sin\theta)^n$.

Euler's formula

Assume that we can write

$$(*) \quad e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Need to find $a_0, a_1, a_2, a_3, \ldots$

Put $x=0$ $\quad 1 = a_0$.

Now differentiate both sides of $(*)$

$$e^x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Put $x=0$ $\quad 1 = a_1$

Now differentiate both sides of the above equ.

$$e^x = 2a_2 + 3\cdot 2\, a_3 x + \cdots$$

Put $x=0$ $\quad 1 = 2a_2 \therefore a_2 = \frac{1}{2}$

Now differentiate both sides of the above eqn

$$e^x = 3 \cdot 2 \cdot a_3 + \dots$$

Put $x = 0$ $\therefore a_3 = \frac{1}{3!}$

etc. We deduce that (with some hand waving)

$$\boxed{e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots}$$

We can do exactly the same thing for $\sin x$ and $\cos x$ (Rem x is in radians)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$