Lecture 13

Last time

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We calculate

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

We use the following table

i	i^2	i^3	i^4	i^5
i	-1	$-i$	1	i

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} i + \frac{\theta^4}{4!} \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) +$$

$$i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$\therefore$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This links sin, cos, exp via i.

Put $\theta = \pi$

Euler's Formula

$$\boxed{e^{i\pi} = -1}$$

We want to calculate cube roots, fourth roots etc of complex numbers. But, before ~~we~~ we can do that, we need to know more about polynomials

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Some material on polynomials follows...

Polynomials

A polynomial of degree n with complex coefficients looks like this

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_n \neq 0$ and $a_0, a_1, \dots, a_n \in \mathbb{C}$. A <u>root of</u> $p(x)$ is a number $r \in \mathbb{C}$ s.t. $p(r) = 0$.

We write $\deg p(x)$ for the degree of $p(x)$ [if $p(x) \equiv 0$, identically zero, then $\deg p(x)$ is not defined]. If $a_n = 1$, the polynomial is said to be <u>monic</u>.

Terminology

degree	name
1	linear
2	quadratic
3	cubic
4	quartic
5	quintic.

<u>Theorem</u> (Not proved here). There are algebraic formulae for the roots of quadratics, cubics, quartics, ~~[illegible]~~ (see my book if you are interested in deriving such formulae). For quintics and beyond, there are <u>no</u> algebraic formulae — a result due to Abel and Galois.

[V. important in the development of algebra].

<u>N.B</u> The above theorem doesn't say polynomials don't have roots, it simply says that you cannot, in general, find them easily.

Our goal now is to explain why finding the roots of a polynomial is a worthwhile occupation.

To do this, we shall need to develop a small part of the theory of polynomials.

Let $f(x)$ and $g(x)$ be polynomials.

If $g(x) = f(x)q(x)$, for some polynomial $q(x)$, we say that $f(x)$ divides $g(x)$ and that $q(x)$ is the quotient. We write $f(x) \mid g(x)$ to mean '$f(x)$ divides $g(x)$'.

If $f(x) \nmid g(x)$ then

remainder

$$g(x) = f(x)q(x) + r(x)$$

where $\deg r(x) < \deg f(x)$. This result is called the remainder theorem (not proved here).

__Remark__ You should know from school how to divide by a linear polynomial. See my book (p181) for general polynomial long division.

__Notation__

$\mathbb{C}[x]$ polynomials with complex coefficients.

$\mathbb{R}[x]$ ———— real ————.

$\mathbb{Q}[x]$ ———— rational ————.

__Lemma__ Let $f(x), g(x)$ be non-zero polys.

Then $f(x)\,g(x) \not\equiv 0$ and

$$\deg f(x)\,g(x) = \deg f(x) + \deg g(x).$$

lower degree

Proof Let $f(x) = a_m x^m + \text{stuff}$

and $g(x) = b_n x^n + \text{stuff}$, where

lower degree.

$a_m, b_n \neq 0$. Then

$$f(x)\,g(x) = a_m b_n x^{m+n} + \text{stuff} \quad \blacksquare$$

The following is the key result.

$$\boxed{\textbf{Lemma} \quad f(a) = 0 \iff (x-a) \mid f(x)}$$

i.e. roots of polynomials correspond exactly to linear factors of the polynomial.