The degree of a polynomial is a measure of how complex the polynomial is. 1

Lecture 14

Lemma $f(a) = 0 \Leftrightarrow (x-a) \mid f(x)$

Proof Easy direction. Suppose $(x-a) \mid f(x)$.

Then $f(x) = q(x)(x-a)$ for some $q(x)$.

Thus $f(a) = q(a)(a-a) = 0$.

It follows that a is a root of $f(x)$.

Hard direction. Suppose that $f(a) = 0$.

There are ~~two~~ possibilities:

(1) $(x-a) \mid f(x)$.

(2) $(x-a) \nmid f(x)$.

We show that (2) cannot happen.

$$f(x) = q(x)(x-a) + r(x) \quad \text{where } \deg r(x) < 1.$$

It follows that $r(x)$ is a constant. Call it r.

Then $f(x) = q(x)(x-a) + r$ (we are assuming $r \neq 0$)

But $f(a) = q(a)\cdot 0 + r = r$. $\therefore r = 0$ - Contradiction ∎

At this point, I can make a very important definition.

<u>Definition</u> Let a be a root of the polynomial $p(x)$. We say that <u>a has multiplicity m</u> if

$$(x-a)^m \mid p(x) \quad \text{but} \quad (x-a)^{m+1} \nmid p(x)$$

This gives a precise meaning to the idea of a repeated root.

We always count roots according to their multiplicities. For example, with this definition a ~~qu~~ quadratic equation <u>ALWAYS</u> has 2 roots.

We now have the following theorem

<u>Theorem</u> A non-constant polynomial of degree n has <u>at most</u> n roots.

($\deg f(x) = n$)

<u>Proof</u> If $f(x)$ has a root a_1 then by our lemma

$$f(x) = (x - a_1) f_1(x) \quad \text{where } \deg f_1(x) = n-1$$

Now repeat with $f_1(x)$ etc.

The max. number of roots that $f(x)$ can have is n, in which case

n roots

$$f(x) = a\,(x - a_1) \ldots\ldots (x - a_n) \quad \blacksquare$$

constant

We now return to studying n^{th} roots of a complex number a $(\neq 0)$. I.e. we are looking for all z s.t. $z^n = a$.

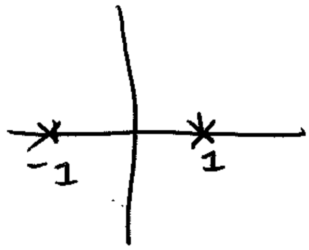
That is all z s.t. $z^n - a = 0$. This means we are looking for the roots of $x^n - a = 0$.

We know there are <u>at most n</u> – we shall prove that there are <u>exactly n</u>.

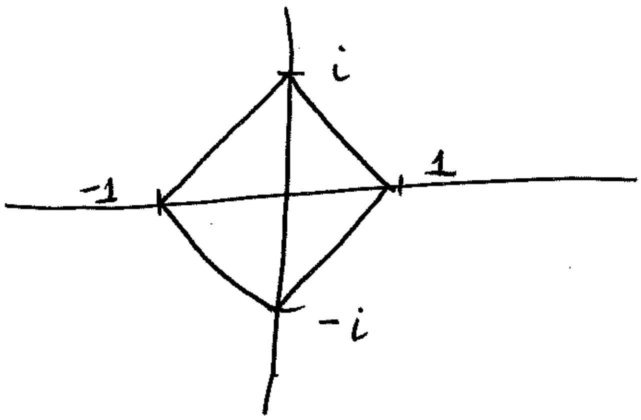
We begin with a special case — the roots of the polynomial $x^n - 1$. We call these roots the n^{th} roots of unity.

Examples

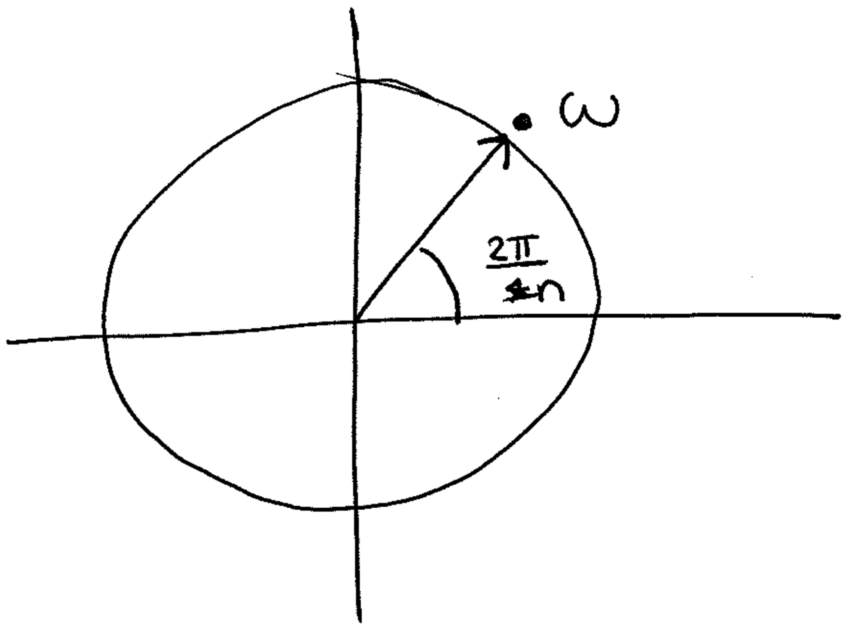
(1) The roots of $x^2 - 1$ are ± 1



(2) The roots of $x^4 - 1$ are $\pm 1, \pm i$



To find all the n^{th} roots of unity, we first locate a special n^{th} root (not 1!) called the principal n^{th} root.



Define $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

By properties of complex multiplication, or de Moivre's theorem, $\omega^n = \cos n\frac{2\pi}{n} + i\sin n\frac{2\pi}{n}$

$= \cos 2\pi + i\sin 2\pi = 1.$

We call ω the principal n^{th} root of unity.

Consider now the following n complex nos.

$$\underbrace{\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}, \omega^n = 1}$$

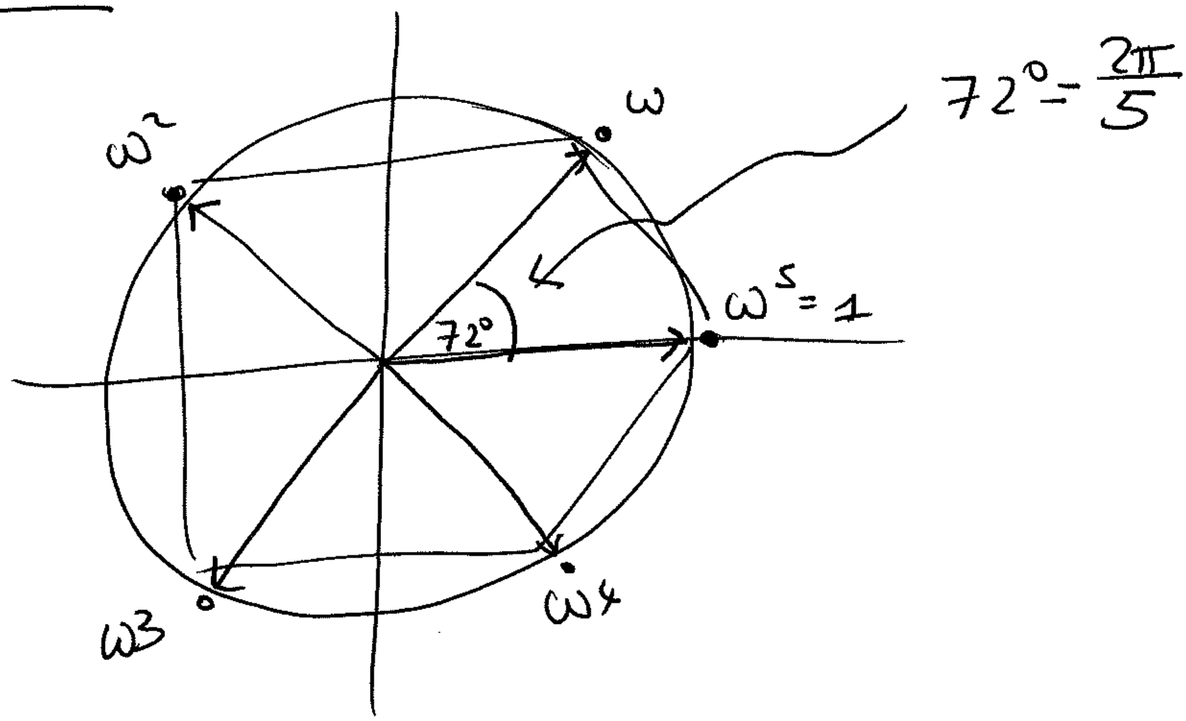
These are all the n^{th} roots of unity.

- let $1 \le j \le n$. Then

$$(\omega^j)^n = \omega^{jn} = (\omega^n)^j = 1^j = 1.$$

- The numbers $\omega, \omega^2, \ldots, \omega^n$ are all distinct since their arguments are all different.

Example Find all the 5^{th} roots of unity.



$1, \omega, \omega^2, \omega^3, \omega^4$ form the vertices of a regular pentagon.

We can use these polar forms to calculate n^{th} roots to any degree of accuracy

$$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$\approx 0.309 + 0.9511\,i$$

$$\omega^2 \approx -0.809 + 0.5878\,i$$

$$\omega^3 \approx -0.809 - 0.5878\,i$$

$$\omega^4 \approx 0.309 - 0.9511\,i$$

$$\omega^5 \approx 1.000 + 0.000\,i \approx 1$$