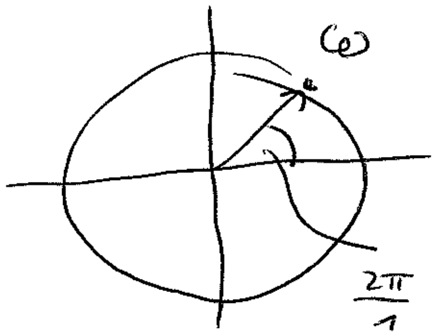
Lecture 15

n^{th} roots of unity Let $n \geq 2$ (really 3).

Put $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$.

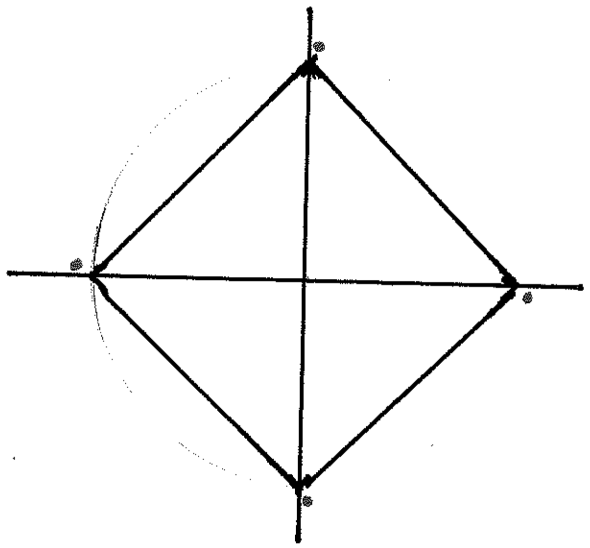


Then ω is called the principal n^{th} root of unity.

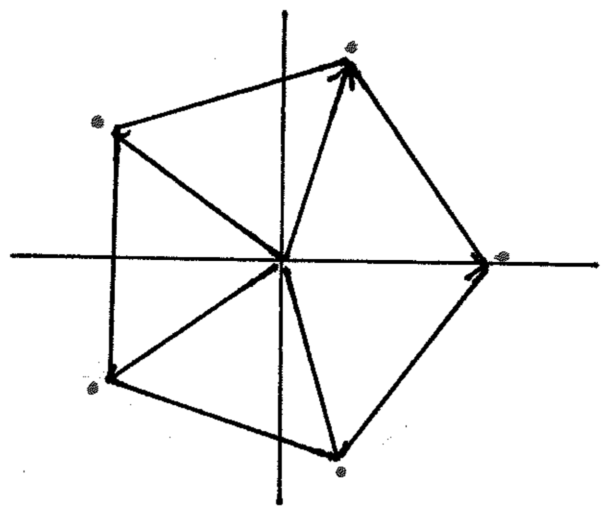
All the n n^{th} roots of unity are then:

$$\underbrace{\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}, \omega^n = 1}_{\text{These ~~are~~ are all the } n^{th} \text{ roots of unity.}}$$

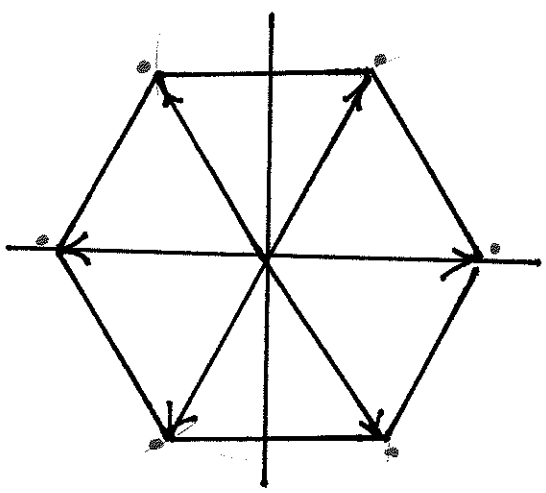
4^{th} roots of unity



5^{th} roots of unity

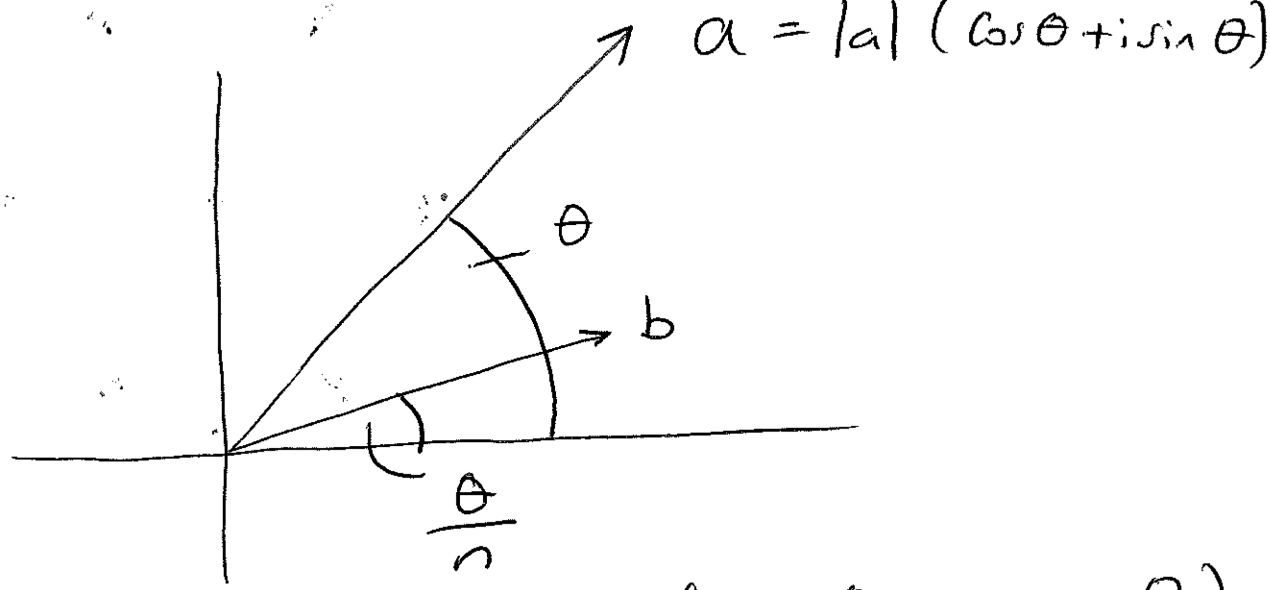


6^{th} roots of unity



We use these results to study n^{th} roots of complex numbers $(n \geq 3)$.

Let a be a complex no. $(a \neq 0)$
An n^{th} root of a is a number z s.t. $z^n = a$.
We can rewrite this as $z^n - a = 0$. This is equivalent to saying that z is a root of the polynomial $x^n - a$.
We know that a non-zero complex no. has <u>at most</u> n, n^{th} roots. I will prove that it has in fact exactly n roots.



Define $b = \sqrt[n]{|a|}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$.

Then $b^n = a$. We call <u>b the principal n^{th} root of a</u>.

<u>Theorem</u> Let $a = |a| (\cos\theta + i \sin\theta)$.

Then the n^{th} roots of a are:

$$\omega b, \omega^2 b, \ldots, \omega^{n-1} b, b$$

where $b = \sqrt[n]{|a|} \left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right)$ where

$$\omega = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}.$$

<u>Example</u> What are the 4^{th} roots of 2.

The principal 4^{th} root is $\sqrt[4]{2}$

Thus the 4^{th} roots of 2 are

$$i\sqrt[4]{2}, \quad -i\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad \sqrt[4]{2}.$$

Since $i, -i, -1, 1$ are the 4^{th} roots of unity.

Radical expressions (not politics)

Roughly speaking, a radical expression is a complex number where n^{th} roots (= radicals) are allowed but no sines or cosines

Example The principal 3^{rd} root of 1 is

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$ (not a radical expression)

but we can also explicitly write down what $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$ in terms of roots to get

$$\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$$ (radical expression).

$$\Big[\ \omega^2 = \frac{1}{4}\left(1 - 2i\sqrt{3} - 3\right)$$

$$= \frac{1}{4}\left(-2 - 2i\sqrt{3}\right)$$

$$= -\frac{1}{2}\left(1 + i\sqrt{3}\right)$$

$$\omega^3 = \frac{1}{2}\left(-1 + i\sqrt{3}\right)\cdot -\frac{1}{2}\left(1 + i\sqrt{3}\right)$$

$$= -\frac{1}{4}\left(-1 - \cancel{i\sqrt{3}} + \cancel{i\sqrt{3}} - 3\right) = \underline{1}\ \Big]$$

The following will not be proved

Theorem (The fundamental theorem of algebra)

Every non-constant poly has a root

Corollary Let $p(x)$ be a poly of degree n.

Then $p(x)$ has exactly n roots and

$$p(x) = a(x-a_1)\dots(x-a_n)$$

$a \in \mathbb{C}$.

$\mathbb{C}[x]$ polys with complex coeffs.

$\mathbb{R}[x]$ polys with real coeffs.

We now apply the above results to *real* polynomials.