Lecture 16

Lemma

(1) $\overline{z_1 + \dots + z_n} = \overline{z_1} + \dots + \overline{z_n}$.

(2) $\overline{z_1 \dots z_n} = \overline{z_1} \dots \overline{z_n}$.

(3) $z \in \mathbb{R}$ iff $z = \overline{z}$.

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set of polys with real coefficients

Lemma Let $p(x) \in \mathbb{R}[x]$.

If z is a root then $\overline{z}$ is a root.

Proof Let $p(x) = a_n x^n + \dots + a_1 x + a_0$ $\left(\text{where } a_i \in \mathbb{R}\right)$

Then $a_n z^n + \dots + a_1 z + a_0 = 0$.

Apply complex conjugation to both sides and apply lemma to get

$$\overline{a_n}\,\overline{z}^n + \dots + \overline{a_1}\,\overline{z} + \overline{a_0} = 0$$

But $a_0, \dots, a_n \in \mathbb{R} \Rightarrow p(\overline{z}) = 0$. ∎

Lemma Let $z \in \mathbb{C} \setminus \mathbb{R}$.

Then $(x-z)(x-\bar{z})$ is a real irreducible quadratic polynomial.

Proof $(x-z)(x-\bar{z}) = x^2 - x\bar{z} - xz + z\bar{z}$

$= x^2 - (\bar{z}+z)x + z\bar{z}$.

Put $z = a+ib$. Then

$\bar{z} + z = (a-ib) + (a+ib) = 2a$.

$z\bar{z} = a^2 + b^2$.

So, $(x-z)(x-\bar{z}) = x^2 - 2ax + (a^2+b^2)$

This is a real quadratic polynomial

Discriminant

$D = (-2a)^2 - 4(a^2+b^2) = 4a^2 - 4a^2 - 4b^2$

$= -4b^2 < 0$ $\blacksquare$

Theorem Fundamental theorem for real poly.

Every real polynomial can be written as a product of real linear and real irreducible quadratic polynomials.

degree n

Proof. Let $p(x) \in \mathbb{R}[x]$.

Divide the n roots of $p(x)$ into real $r_1, \ldots, r_s$ and complex $z_1, \ldots, z_t$.

But, the complex roots come in complex conjugate pairs $u_1, \bar{u}_1, \ldots, u_m \bar{u}_m$.

$$\therefore \quad p(x) = a\,(x-r_1)\ldots(x-r_s)\,\overbrace{(x-u_1)(x-\bar{u}_1)}\ldots\overbrace{(x-u_m)(x-\bar{u}_m)}$$

real const.

but $(x-u_i)(x-\bar{u}_i)$ multiplies out

to give a real irreducible quadratic ∎

Example

Let $p(x) = x^4 - 3x^3 + 3x^2 - 3x + 2$.

This is a real polynomial. We will write it as a product of real linear and real irreducible quadratic poly. We find that 1 and 2 are roots.

Thus $p(x) = (x-1)(x-2)(x^2+1)$

x^2+1 is a real irreducible quadratic since $D^2 < 0$.

How to find roots

(1) If $p(x) \in \mathbb{R}[x]$ and you are given a root $u \in \mathbb{C} \setminus \mathbb{R}$ then $\overline{u}$ will also be a root.

(2) Let $p(x)$ be monic with integer coefficients. Any integer root must divide the constant term of $p(x)$.

Proof.

$$p(x) = \underbrace{x^n}_{\text{monic}} + a_{n-1}x^{n-1} + \dots + a_1 x + \overbrace{a_0}^{\text{constant term}}$$

$a_{n-1}, \dots, a_0 \in \mathbb{Z}$.

Let $s \in \mathbb{Z}$ be a root. Then

$$s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0.$$

So $\quad a_0 = -s^n - a_{n-1}s^{n-1} - \dots - a_1 s$

$$= s\underbrace{\left[-s^{n-1} - a_{n-1}s^{n-2} - \dots - a_1\right]}_{\text{integer.}}$$

So $\quad s \mid a_0$.

END

OF

SECTION 2