Lecture 17

Section 3

Definition A Matrix is a rectangular array of numbers (usually, but other mathematical quantities also can occur)

$$\begin{array}{c c} & \text{4 columns} \\ \text{3 rows} & \begin{pmatrix} 3 & 4 & 6 & 7 \\ 8 & 0 & -\pi & e \\ \sqrt{2} & 1 & i & -i \end{pmatrix} \end{array} \qquad 3 \times 4 \text{ Matrix}$$

In general, an $m \times n$ matrix has m rows and n columns ~~and~~ $m \times n$ is called the size of the matrix. Matrices are usually denoted by uppercase Latin letters: $A, B, C, \ldots$

$(A)_{ij}$ = element of matrix A in row i and column j

(also A_{ij})

<u>Example</u> If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$

Then $(A)_{23} = 7$

<u>Definition</u> Matrices A and B are <u>equal</u>

if (1) they have the same size and
(2) corresponding entries/elements are equal i.e. $(A)_{ij} = (B)_{ij}$ for all appropriate values of i and j.

1. <u>Scalars</u> In matrix theory, numbers tend to be called <u>scalars</u> and denoted by l.c. Greek letters: λ, μ, etc.

Matrix arithmetic

Addition Matrices A and B can only be added if they have the same size (if they don't addition is not defined). Then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

Example $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0+1 & 1+2 \\ -1+2 & 2+1 \end{pmatrix}$

$$= \underline{\underline{\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}}}$$

Subtraction If A and B have the same size

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij}.$$

Scalar multiplication $(\lambda A)_{ij} = \lambda (A)_{ij}$

Exple

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

The transpose of A

$$(A^T)_{ij} = (A)_{ji}$$

Exple $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$

Multiplication We need some preparation.

$(a_1\ a_2 \dots a_m)$ is called a ~~row matrix~~ row vector

$\begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ a_m \end{pmatrix}$ is called a ~~Column matrix~~ Column vector

The inner product of a row matrix and a column matrix

let $\underline{a} = (a_1 \dots a_m)$

$$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

inner product

The $\underline{a} \cdot \underline{b}$ is only defined if $m = n$

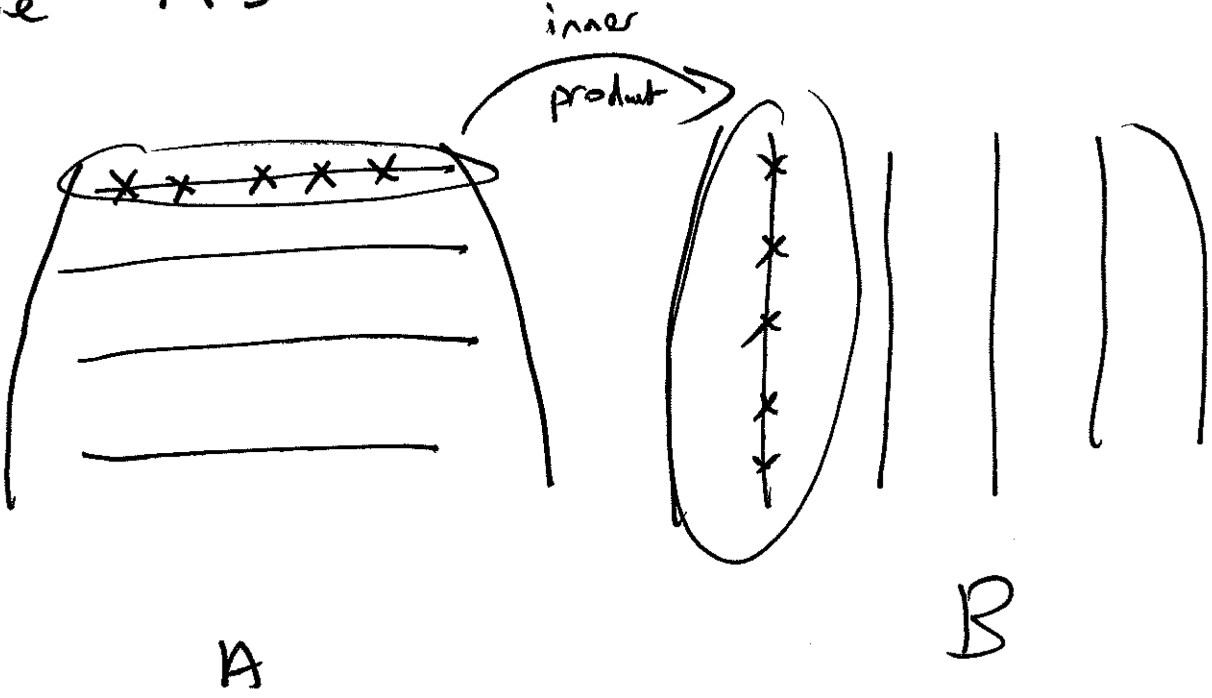
In which case

$$\underline{a} \cdot \underline{b} = \overbrace{a_1 b_1 + \dots + a_m b_m}^{\text{a single number}}$$

$$= \sum_{i=1}^{m} a_i b_i$$

Matrix multiplication

This is more complex than the other matrix operations, but very important. Want to define AB



Think of A as being a sequence of row matrices

Think of B as being a sequence of column matrices

Need for AB to be defined:

columns of A = # rows of B
(else not defined)

$$\underset{\overset{\shortparallel}{A}}{\begin{pmatrix} \longleftarrow \ \underline{a}_1 \ \longrightarrow \\ \vdots \\ \longleftarrow \ \underline{a}_m \ \longrightarrow \end{pmatrix}} \underset{\overset{\shortparallel}{B}}{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}$$

$$= \underset{\overset{\shortparallel}{AB}}{\begin{pmatrix} \underline{a}_1 \cdot \underline{b}_1 & \underline{a}_1 \cdot \underline{b}_2 & \cdots & \underline{a}_1 \cdot \underline{b}_n \\ & \vdots & & \\ \underline{a}_m \cdot \underline{b}_1 & \underline{a}_m \cdot \underline{b}_2 & \cdots & \underline{a}_m \cdot \underline{b}_n \end{pmatrix}}$$

Example

$$\underbrace{2 \times \overbrace{3 \cdot 3}^{=} \times 2}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1\ 2\ 1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & (1\ 2\ 1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ (0\ 1\ -1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & (0\ 1\ -1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2 + 0 & 2 - 2 + 1 \\ 0 + 1 + 0 & 0 - 1 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

If A is $m \times p$ and B is $p \times n$

then AB is $m \times n$.

As usual, $A^2 = AA$

$$A^3 = (AA)A$$

if defined.