Lecture 18

Matrix algebra

matrix addition

(MA1) $(A+B)+C = (A+B)+C$.

This is the associative law for matrix addition.

(MA2) $A + O = A = O + A$. Here O is the zero matrix of the same size of A called the additive identity of A.

(MA3) $A + (-A) = O = (-A) + A$.

Where $-A = -1A$, the additive inverse of A.

(MA4) $A + B = B + A$.

Matrix addition is commutative.

Matrix multiplication

(MM1) $(AB)C = A(BC)$ when the products are defined. This is the associative law for multiplication.

(MM2) $IA = A = AI$
where I is the identity matrix of the appropriate size
$I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, ...
We usually just write I rather than I_n.
A matrix is square if # rows = # columns.
For non-square matrices, I will be different
the LHS from the RHS.

(MM3) $$\left.\begin{aligned} A(B+C) &= AB + AC \\ (B+C)A &= BA + CA \end{aligned}\right\} \text{ distributive laws.}$$

Properties of scalar multiplication

(S1) $1A = A, \quad -1A = -A.$

(S2) $0A = 0.$

(S3) $\lambda(A+B) = \lambda A + \lambda B.$

λ, μ scalars.

(S4) $(\lambda\mu)A = \lambda(\mu A).$

(S5) $(\lambda+\mu)A = \lambda A + \mu A.$

(S6) $(\lambda A)B = A(\lambda B) = \lambda(AB)$

Properties of the transpose

(T1) $(A^T)^T = A.$

(T2) $(A+B)^T = A^T + B^T.$

(T3) $(\lambda A)^T = \lambda A^T.$

(T4) $(AB)^T = B^T A^T \leftarrow$ NB

Counterexamples

(1) Matrix multiplication is not commutative.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 7 \end{pmatrix}$$

$$\neq$$

$$BA = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}$$

(2) Two non-zero matrices can multiply to zero.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

All of these properties of matrices can be proved from the definitions. I will give only one example of such a proof to provide a flavor.

Proof to the associativity of matrix multiplication

Need to prove that $(AB)C = A(BC)$ when defined.

Recall: def'n of equality of matrices

(1) Need to show that $(AB)C$ and $A(BC)$ have the same size.

$$\begin{matrix} A & B & C \\ m\times p & p\times n & n\times q \end{matrix}$$

Size of $(AB)C$ is $m\times q$

Size of $A(BC)$ is $m\times q$

(2) We now prove that

$$((AB)C)_{ij} = (A(BC))_{ij}$$

for all appropriate values of i and j.

<u>Convention</u> $(X)_{ij} = x_{ij}$.

$$(XY)_{ij} = \sum_{k} x_{ik} y_{kj}$$

$$((AB)C)_{ij} = \sum_k (AB)_{ik} (C)_{kj}$$

$$= \sum_k \left(\sum_l (A)_{il} (B)_{lk} \right) (C)_{kj}$$

$$= \sum_k \left(\sum_l a_{il} b_{lk} \right) c_{kj}$$

distributivity of scalars

$$= \sum_k \sum_l a_{il} b_{lk} c_{kj} \quad (*)$$

$$(A(BC))_{ij} = \sum_l (A)_{il} (BC)_{lj}$$

$$= \sum_l (A)_{il} \left(\sum_k (B)_{lk} (C)_{kj} \right)$$

$$= \sum_l a_{il} \left(\sum_k b_{lk} c_{kj} \right)$$

$$= \sum_l \sum_k a_{il} b_{lk} c_{kj} \quad (D)$$

(*) and (D) are equal, since we are just adding up the same numbers in a different order. ∎