

Lecture 19

Linear equations

Example Consider the following system of three linear equations in four unknowns:

$$\begin{cases} 3x - 2y + z + 6t = 5 \\ x + 3y - z + 5t = 4 \\ -2x - y - z - 2t = 3 \end{cases}$$

This can be written in matrix form

explains form
of matrix
multiplication

$$\underbrace{\begin{pmatrix} 3 & -2 & 1 & 6 \\ 1 & 3 & -1 & 5 \\ -2 & -1 & -1 & -2 \end{pmatrix}}_{\text{matrix of coefficients}} \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{\text{unknown}} = \underbrace{\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}}_{\text{given}}$$

This has to be $A \underline{x} = \underline{b}$ where

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A \text{ is } m \times n \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Any column matrix $\underline{a}$ st. $A \underline{a} = \underline{b}$ is called a solution. The solution set of

$A \underline{a} = \underline{b}$ is the set

$$\{ \underline{a} : A \underline{a} = \underline{b} \}$$

We shall now use the matrix properties to prove our first theorem

Theorem (Fundamental theorem of linear equations)

Let $A\underline{x} = \underline{b}$ be a system of linear equations (over $\mathbb{C}$). Then there are 3 possibilities.

(1) There are no solutions. } Inconsistent.

(2) Exactly one solution. } consistent

(3) Infinitely many solutions. }

Proof Assume that there are at least two solutions $\underline{u}$ and $\underline{v}$ ($\underline{u} \neq \underline{v}$). So, $A\underline{u} = \underline{b}$ and $A\underline{v} = \underline{b}$.

$$\text{Then } A(\underline{u} - \underline{v}) = A\underline{u} - A\underline{v} = \underline{b} - \underline{b} = \underline{0}$$

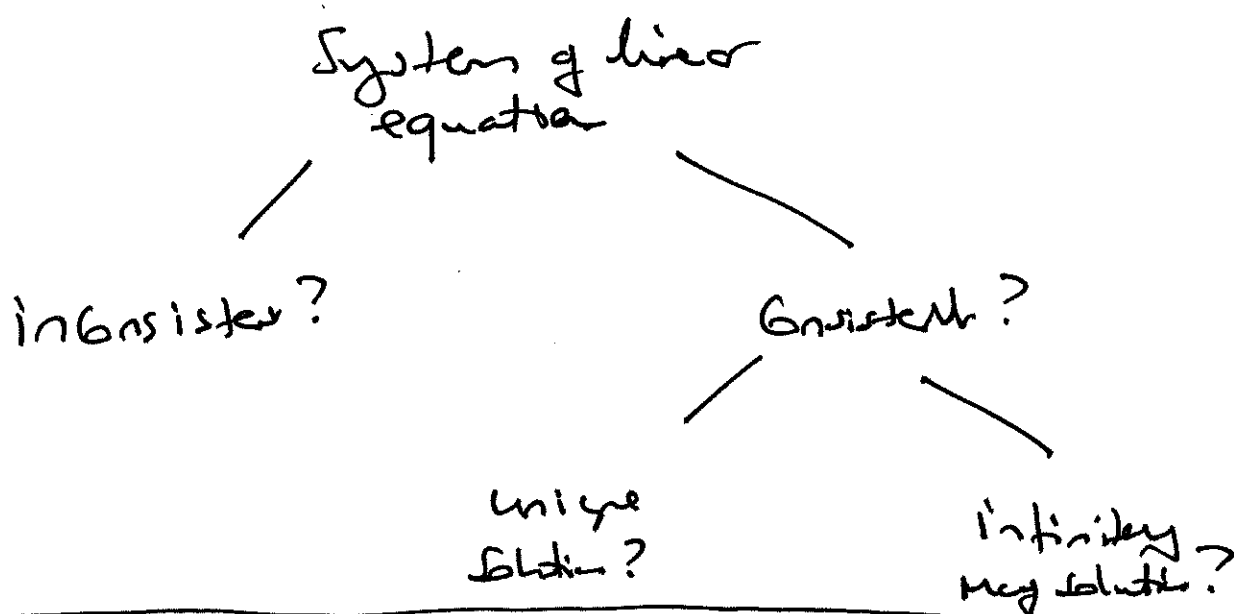
Let λ be any scalar. Then

$$A(\lambda(\underline{u} - \underline{v}) + \underline{u}) = \lambda A(\underline{u} - \underline{v}) + A\underline{u} \\ = \underline{0} + \underline{b} = \underline{b}$$

$\therefore$ For all $\lambda \in \mathbb{C}$,

$$\lambda(\underline{u} - \underline{v}) + \underline{u}$$

are solutions. We have therefore found infinitely many solutions. ■



We shall now develop ~~an~~ an algorithm to answer these questions.