Lecture 19

Linear equations

Example Consider the following system of three linear equations in four unknowns:

$$\begin{cases} 3x - 2y + z + 6t = 5 \\ x + 8y - z + 5t = 4 \\ -2x - y - z - 2t = 3 \end{cases}$$

This can be written in matrix form

explains form of matrix multiplication

$$\underbrace{\begin{pmatrix} 3 & -2 & 1 & 6 \\ 1 & 3 & -1 & 5 \\ -2 & -1 & -1 & -2 \end{pmatrix}}_{\text{matrix of coefficients}} \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{\text{unknown}} = \underbrace{\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}}_{\text{given}}$$

This has the form $A\underline{x} = \underline{b}$ where

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A \text{ is } m \times n \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Any column matrix $\underline{a}$ s.t. $A\underline{a} = \underline{b}$ is called a <u>solution</u>. The <u>solution set</u> of $A\underline{x} = \underline{b}$ is the set

$$\{\underline{a} : A\underline{a} = \underline{b}\}$$

We shall now use the matrix properties to prove our first theorem

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Theorem (Fundamental theorem of linear equations)

Let $A\underline{x} = \underline{b}$ be a system of linear equations (over $\mathbb{C}$). Then there are 3 possibilities.

(1) There are no solutions. } Inconsistent.

(2) Exactly one solution.

(3) Infinitely many solutions.

} Consistent

Proof Assume that there are at least two solutions $\underline{u}$ and $\underline{v}$ ($\underline{u} \neq \underline{v}$). So, $A\underline{u} = \underline{b}$ and $A\underline{v} = \underline{b}$.

Then $A(\underline{u}-\underline{v}) = A\underline{u} - A\underline{v} = \underline{b} - \underline{b} = \underline{0}$

Let λ be any scalar. Then

$$A\left(\lambda(\underline{u}-\underline{v})+\underline{u}\right) = \lambda A(\underline{u}-\underline{v}) + A\underline{u}$$
$$= \underline{0} + \underline{b} = \underline{b}$$

$\therefore$ for all $\lambda \in \mathbb{C}$,

$\lambda(\underline{u}-\underline{v}) + \underline{u}$

are solutions. We have therefore found infinitely many solutions. $\blacksquare$

System of linear equations

- inconsistent?
- consistent?
  - unique solution?
  - infinitely many solutions?

$\boxed{\text{We shall now develop an \underline{algorithm} to answer these questions.}}$