Lecture 20

Algorithm for solving linear equations

Let $A\underline{x} = \underline{b}$ be our system of linear equations. We will operate with the augmented matrix $(A|\underline{b})$. We shall apply what are known as elementary row operations (EROs) to the augmented matrix $(A|\underline{b})$. This will yield a new augmented matrix matrix $(A'|\underline{b}')$ which has a shape making it an echelon matrix. The equations associated with $(A'|\underline{b}')$ are

(1) Easy to solve

(2) Have the same solutions as the original set of equations

$A\underline{x} = \underline{b}$ ← same solution set → $A'\underline{x} = \underline{b}'$ easy to solve & have same solutions as $A\underline{x} = \underline{b}$

$$(A|\underline{b}) \xrightarrow[\text{ERO's}]{} (A'|\underline{b}')$$

← defined later

echelon matrix

Elementary row operations

1. $R_i \leftarrow \lambda R_i$ $(\lambda \neq 0)$

   multiply row i by a non-zero scalar.

2. $R_i \leftrightarrow R_j$.

   Interchange row i and row j

3. $R_j \leftarrow R_j + \lambda R_i$

   Add λ times row i to row j

__Proposition.__ Let $A\underline{x} = \underline{b}$ be a system of linear equations. Apply ERO to $(A|\underline{b})$ to obtain $(A'|\underline{b}')$. Then the solution set of $A\underline{x} = \underline{b}$ is the same as the solution set of $A'\underline{x} = \underline{b}'$.

$\boxed{\text{Thus EROs change the equations BUT do not change the solutions}}$

Example

System of linear equations.

$$\begin{cases} x + 2y - 3z = -1 \\ 3x - y + 2z = 7 \\ 5x + 3y - 4z = 2 \end{cases}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

$$\downarrow \; R_2 \leftarrow R_2 - 3R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

Calculations

$$\begin{array}{cccc} -3 & -6 & 9 & 3 \\ 3 & -1 & 2 & 7 \\ \hline 0 & -7 & 11 & 10 \end{array}$$

$R_3 \leftarrow R_3 - 5R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right)$$

$R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$$\begin{array}{cccc} -5 & -10 & 15 & 5 \\ 5 & 3 & -4 & 2 \\ \hline 0 & -7 & 11 & 7 \end{array}$$

The equations associated with this augmented matrix are

$$x + 2y - 3z = -1$$

$$-7y + 11z = 10$$

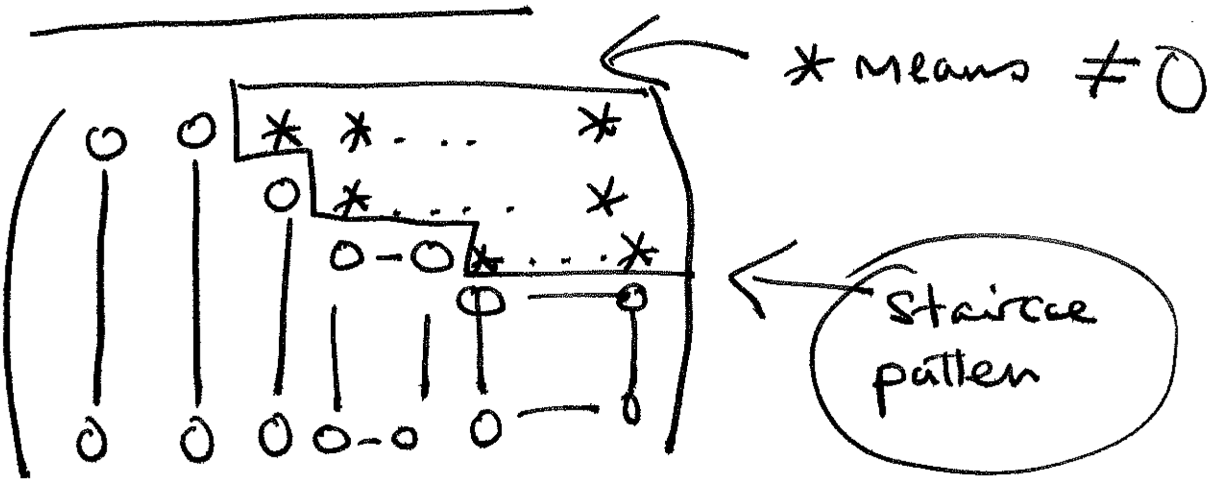
$$\boxed{0x + 0y + 0z = -3}$$

$0 = -3$ # NO SOLUTIONS

It follows that the original system of equations is inconsistent. $\square$

We shall next describe <u>echelon matrices</u> and then describe the algorithm for solving systems of linear equations.

The matrix $(A' \mid \underline{b}')$ has to be an <u>echelon matrix</u>. These look like this



<u>Special case</u> When A is 3×3.

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right)$$

Use top left non-zero entry to produce 0s below it using EROs

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right)$$

Now repeat with leading non-zero entry in next row.

Various patterns can arise:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right) \leftarrow \text{inconsistent}$$

We have already seen an example of this.

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

Consistent

unique solution

(I will show you an example later)

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \text{Consistent}$$

infinitely many solutions

or

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$