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Lecture 21

Examples These illustrate the three kinds of ~~solutions~~ outcomes in the Fundamental Theorem of Linear equations.

1. We show that the following system is inconsistent (= no solution)

I did this last time

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

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$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

$R_2 \leftarrow R_2 - 3R_1$

-3	-6	9	3
3	-1	2	7
0	-7	11	10

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

-5	-10	15	5
5	3	-4	2
0	-7	11	7

$R_3 \leftarrow R_3 - 5R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right)$$

$R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$0x + 0y + 0z = 3$

$0 = -3$

inconsistent

staircase pattern

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ \underbrace{0 \quad 0 \quad 0}_{\text{all 0s}} & & & \underbrace{-3}_{\text{non-zero}} \end{array}\right)$$

inconsistent.

2. <u>We show that the following system has exactly one solution</u>

<u>System of equations.</u>

$$x + 2y + 3z = 4$$

$$2x + 2y + 4z = 0$$

$$3x + 4y + 5z = 2$$

<u>Augmented matrix</u>

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 3 & 4 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & -2 & -4 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Unique solution

$R_2 \leftarrow R_2 - 2R_1$

-2	-4	-6	-8
2	2	4	0
0	-2	-2	-8

$R_3 \leftarrow R_3 - 3R_1$

-3	-6	-9	-12
3	4	5	2
0	-2	-4	-10

$R_3 \leftarrow R_3 - R_2$

0	-2	-4	-10
0	2	2	8
0	0	-2	-2

$R_2 \leftarrow -\frac{1}{2} R_2$

$R_2 \leftarrow -\frac{1}{2} R_3$

$$x + 2y + 3z = 4$$
$$y + z = 4$$
$$z = 1$$

back substitute.

$$y = 4 - z = 4 - 1 = 3$$

$$x = 4 - 2y - 3z$$
$$= 4 - 6 - 3 = -5$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$

YOU MUST CHECK SOLUTION WORKS IN ALL EQUATIONS!

<u>Check</u>

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -5+6+3 = 4 \\ -10+6+4 = 0 \\ -15+12+5 = 2 \end{pmatrix}$$

This is how you check answers.

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3. We solve a system where there are infinitely many solutions.

$$\left.\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}\right\} \text{ equations.}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right)$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array}\right)$$

$\downarrow$ $R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

WHOLE ROW IS ZERO

$R_2 \leftarrow R_2 - 2R_1$

$$\begin{array}{cccc} -2 & -4 & 6 & -12 \\ 2 & -1 & 4 & 2 \\ \hline 0 & -5 & 10 & -10 \end{array}$$

$R_3 \leftarrow R_3 - 4R_1$

$$\begin{array}{cccc} -4 & -8 & 12 & -24 \\ 4 & 3 & -2 & 14 \\ \hline 0 & -5 & 10 & -10 \end{array}$$

Q7

$R_2 \leftarrow -\frac{1}{5} R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ MOS so Consistent.

$$x + 2y - 3z = 6$$
$$y - 2z = 2$$

One free variable

<u>Put $z = \lambda$ any scalar</u> $\quad y = 2 + 2\lambda$

$$x = 6 - 2y + 3z$$
$$= 6 - 2(2 + 2\lambda) + 3\cancel{z}\lambda$$
$$= 6 - 4 - 4\lambda + 3\cancel{z}\lambda$$
$$= 2 - \lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - \lambda \\ 2 + 2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

<u>Check</u>

YOU MUST CHECK ALL EQUATIONS

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2-\lambda \\ 2+2\lambda \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2 - \lambda + 4 + \lambda 4 - 3\lambda = 6 \\ 4 - \cancel{2\lambda} - 2 - \cancel{2\lambda} + \cancel{4\lambda} = 2 \\ 8 - \cancel{4\lambda} + 6 + \cancel{6\lambda} - \cancel{2\lambda} = 14 \end{pmatrix}$$

This is how you check solutions.

<u>Example</u> A system such as

$$x + 2y + 3z = 4$$

will have <u>two free variables</u>. Put $z = \lambda$ and $y = \mu$

Then $x = 4 - 2\mu - 3\lambda$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\mu - 3\lambda \\ \mu \\ \lambda \end{pmatrix}$$ is the general solution where $\lambda, \mu \in \mathbb{C}$

Inverses

We now turn to deal with the case where we can divide one matrix by another. This requires an auxiliary notion. We introduce it purely algebraically here and then provide a geometric interpretation in Section 4.

<u>The determinant</u>

<u>Definition</u> Let A be a <u>square matrix</u>.

We define the <u>determinant</u> of A, written $\det(A)$

or $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{vmatrix}$, as follows.

$$\det(a) = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \underline{\underline{a}}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \underline{\underline{b}}\begin{vmatrix} d & f \\ g & i \end{vmatrix} + \underline{\underline{c}}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

¡ Pay attention to <u>signs</u> !

Exaple

(1) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4$

$= 10 - 12 = \underline{\underline{-2}}$

(2) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$

$+ 1 \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}$

$= 1 - 2(3) + (-2)$

$= 1 - 6 - 2 = \underline{\underline{-7}}$

The key property we shall need in this section is the following. I shall not prove it in generality but the proof in the 2×2 case can be found on pp 249 - 250 of my book.

Theorem Let A and B be $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.