Lecture 22

Back to inverses

Definition Let A be a square matrix. We say that A is **invertible** if there is a square matrix B s.t.

$$AB = \underset{\substack{\uparrow \\ \text{identity matrix same size as } A}}{I} = BA$$

We call B an **inverse** of A.

Lemma Let A be an invertible matrix. S'pose that $AB = I = BA$ and $AC = I = CA$. Then $B = C$.

Proof $AB = I$

so, $C(AB) = CI = C$

$$\begin{aligned} C(AB) &= (CA)B \quad \text{by associativity} \\ &= IB \\ &= B. \end{aligned}$$

It follows that $B = C$. ∎

Because of the above lemma, if a matrix A has an inverse it has a unique inverse. We denote this unique inverse (when we know it exists) by A^{-1}.

Example Let $A\underline{x} = \underline{b}$ where A is invertible.
Then $A^{-1}(A\underline{x}) = A^{-1}\underline{b}$. But $A^{-1}(A\underline{x}) = (A^{-1}A)\underline{x} = I\underline{x} = \underline{x}$.
It follows that $\boxed{\underline{x} = A^{-1}\underline{b}}$

Observe that $\underline{b}$ can vary but we can still easily solve the new equations by using A^{-1} (rather than going back to square one).

We would like to answer the following two questions:
(1) How do we decide whether a matrix is invertible or not?
(2) If a matrix is invertible, how do we compute its inverse.

Lemma If A is invertible then $\det(A) \neq 0$.

Proof By definition, $AA^{-1} = I$.
Thus $\det(AA^{-1}) = \det(I) = 1$.
Therefore $\det(A)\det(A^{-1}) = 1$.
It follows that $\det(A) \neq 0$. ∎

In fact,

$$\boxed{A \text{ is invertible} \iff \det(A) \neq 0}$$

Proved in the 2×2 and 3×3 cases.

2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We construct a matrix called the adjugate of A, written $\text{adj}(A)$, with the property that

$$A\,\text{adj}(A) = \det(A) I = \text{adj}(A)\, A.$$

Input $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Procedure

- For each entry, cross out the row and column containing that entry and write down the number you see

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

- Add signs $(-1)^{i+j}$ where i is the row number and j is the column number

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

- TAKE THE TRANSPOSE

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \text{adj}(A)$$

Check

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =$$

$$\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \det(A)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that if $\det(A) \neq 0$ then

$$\boxed{A^{-1} = \frac{1}{\det(A)}\, \text{adj}(A)}$$

3 x 3 case

This is identical to the 2×2 case except that in the first step, for each entry, cross out its row and column and write down the <u>determinant</u> of the matrix you see. The remaining steps are the same.

<u>Example</u> We compute the adjugate of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

- $$\begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 5 & 2 \\ 1 & 5 & 3 \\ 2 & -5 & -4 \end{pmatrix}$$

- Now multiply entry in row i and column j by $(-1)^{i+j}$.

$$\begin{pmatrix} -1 & -5 & 2 \\ -1 & 5 & -3 \\ 2 & 5 & -4 \end{pmatrix}$$

- Now take the transpose

$$\begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix} = \operatorname{adj}(A)$$

<u>Check</u>

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}$$

$$= -5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}^{-1} = \frac{-1}{5} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}$$

$$\boxed{\begin{array}{c}\text{In general, if } \det(A) \neq 0 \text{ then} \\ A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)\end{array}}$$