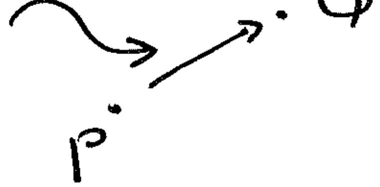
Lecture 23

Section 4: Vectors

We first view vectors <u>geometrically</u> and then <u>algebraically</u>. $\boxed{\text{See also book}}$

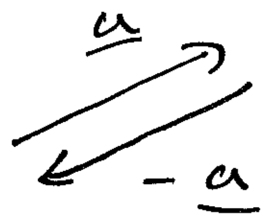
<u>Definition</u> A <u>vector</u> is a directed line segment (in space) which can be moved parallel to itself. Vectors are denoted by bold letters $\underline{a}, \underline{b}, \underline{c}, \ldots$

If P and Q are two points in space then $\overrightarrow{PQ}$ denotes the vector determined by the directed line segment

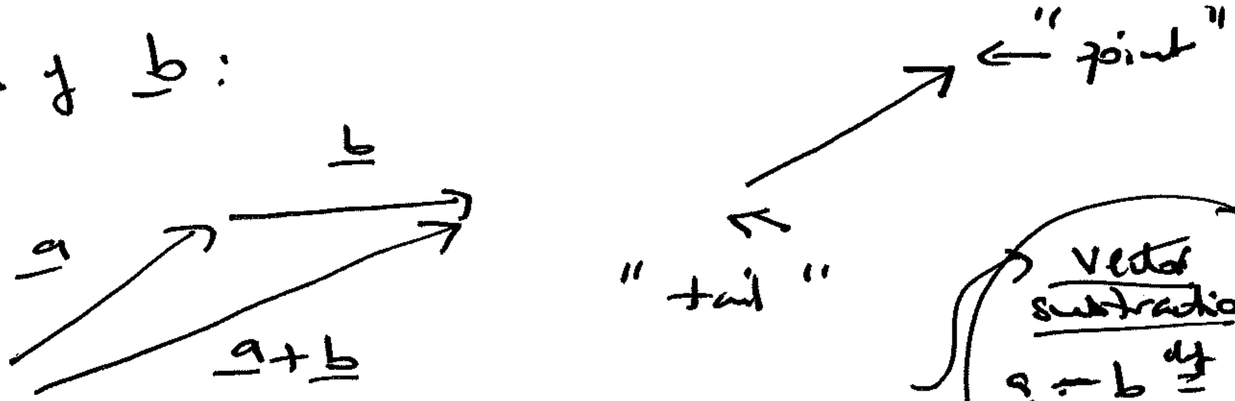


$\overrightarrow{PP}$: The directed line segment determined by P and P is just a point. This determines the zero vector $\underline{0}$.

If $\underline{a}$ is a vector, then $-\underline{a}$ is the vector pointing in the opposite direction



If $\underline{a}$ & $\underline{b}$ are vectors then we may add them — this may require $\underline{a}$ to be moved (or $\underline{b}$!) so that the point of $\underline{a}$ touches the tail of $\underline{b}$:



Vector subtraction

$\underline{a} - \underline{b} \stackrel{\text{def}}{=} \underline{a} + (-\underline{b})$

Properties of vector addition

1. $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$.
2. $\underline{0} + \underline{a} = \underline{a} = \underline{a} + \underline{0}$.
3. $\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$.
4. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$.

Let $\underline{a}$ be a vector.

Define $\|\underline{a}\|$, the <u>length</u> of a.

$\|\underline{a}\| \geq 0$ and $\|\underline{a}\| = 0 \iff \underline{a} = \underline{0}$.

Can multiply a vector $\underline{a}$ by a ~~scal~~ scalar $\lambda (\in \mathbb{R})$ to get a vector $\lambda \underline{a}$.

- If $\lambda > 0$ then $\lambda \underline{a}$ has the same direction as $\underline{a}$ but length $\lambda \|\underline{a}\|$.
- If $\lambda = 0$ then $0 \cdot \underline{a} = \underline{0}$, the zero vector.
- If $\lambda < 0$ then $\lambda \underline{a}$ has direction $-\underline{a}$ and length $|\lambda| \, \|\underline{a}\|$.

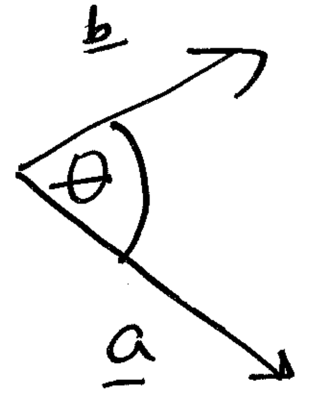
If $\underline{a} \neq \underline{0}$ then $\hat{\underline{a}} = \frac{1}{\|\underline{a}\|} \underline{a}$ is a <u>unit vector</u> (vector of length 1) pointing in the same direction as $\underline{a}$.

Properties of scalar multiplication

1. $0\,\underline{a} = \underline{0}$.
2. $1\,\underline{a} = \underline{a}$. $\quad (\lambda, \mu \in \mathbb{R})$
3. $(-1)\,\underline{a} = -\underline{a}$.
4. $(\lambda+\mu)\,\underline{a} = \lambda\underline{a} + \mu\underline{a}$.
5. $\lambda(\underline{a}+\underline{b}) = \lambda\underline{a} + \lambda\underline{b}$.
6. $\lambda(\mu\underline{a}) = (\lambda\mu)\,\underline{a}$

Operations on vectors

If $\underline{a}, \underline{b} \neq \underline{0}$ denote the angle between them by θ $(0 \le \theta \le \pi)$



<u>Define</u> $\underline{a} \cdot \underline{b} = \|\underline{a}\| \, \|\underline{b}\| \cos\theta$

If $\underline{a}$ or $\underline{b}$ is $\underline{0}$ then define $\underline{a} \cdot \underline{b} = 0$.

This is called the <u>inner product</u> of two vectors.

<u>Properties of the inner product</u>

1. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.
2. $\underline{a} \cdot \underline{a} = \|\underline{a}\|^2$.
3. $\lambda(\underline{a} \cdot \underline{b}) = (\lambda \underline{a}) \cdot \underline{b} = \underline{a} \cdot (\lambda \underline{b})$

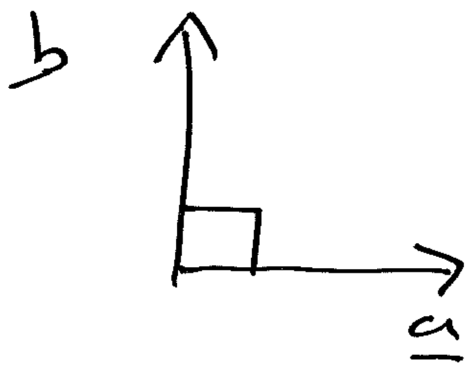
when $\lambda \in \mathbb{R}$.

4. $\boxed{\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}}$

See book for proof.

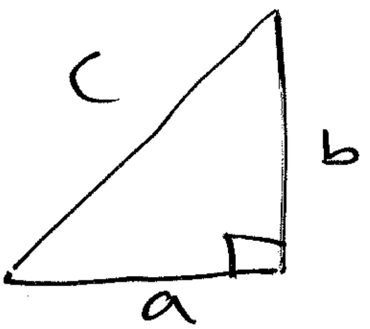
If $\underline{a}, \underline{b} \neq \underline{0}$ and $\underline{a} \cdot \underline{b} = 0$

we say that $\underline{a}$ and $\underline{b}$ are <u>orthogonal</u>.



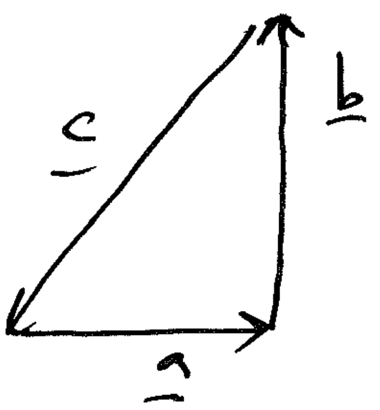
$\underline{a}$ and $\underline{b}$ are <u>ortogonal</u>

Application: proof of Pythagoras' theorem



$a^2 + b^2 = c.$

Choose vectors as shown



where $\|\underline{a}\| = a$,
$\|\underline{b}\| = b$,
$\|\underline{c}\| = c$.

Observe $\underline{a} + \underline{b} + \underline{c} = \underline{0}$

$\Rightarrow \underline{a} + \underline{b} = -\underline{c}$.

Take inner products of both sides

$(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = \underline{c} \cdot \underline{c} = \|\underline{c}\|^2$

$\|$

$\underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b}$

But $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ & $\underline{a} \cdot \underline{b} = 0$

Thus $\|\underline{a}\|^2 + \|\underline{b}\|^2 = \|\underline{c}\|^2$

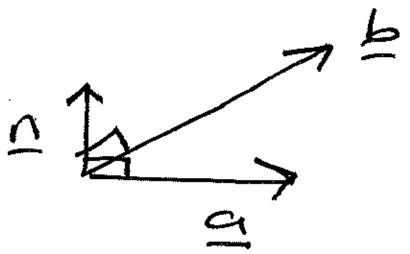
ie $a^2 + b^2 = c^2$ $\blacksquare$

Let $\underline{a}$ and $\underline{b}$ be non-zero vectors.

area = $\|\underline{b}\| \sin\theta \, \|\underline{a}\|$

$$\underline{a} \times \underline{b} = \underbrace{\|\underline{a}\| \, \|\underline{b}\| \sin\theta}_{\text{scalar}} \; \underbrace{\underline{n}}_{\text{vector}}$$

Where $\underline{n}$ is a unit vector ~~+~~ orthogonal to both $\underline{a}$ and $\underline{b}$ where $\underline{n}$ is chosen as below



If either $\underline{a}$ or $\underline{b}$ is $\underline{0}$ we define $\underline{a} \times \underline{b} = \underline{0}$.

We call $\underline{a} \times \underline{b}$ the <u>vector product of $\underline{a}$ and $\underline{b}$</u>.

<u>Properties of the vector product</u>

1. $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.
2. $\lambda(\underline{a} \times \underline{b}) = (\lambda \underline{a}) \times \underline{b} = \underline{a} \times (\lambda \underline{b})$ where $\lambda \in \mathbb{R}$
3. $\boxed{\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}}$. proved in book

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<u>NB</u> Vector product is not associative.