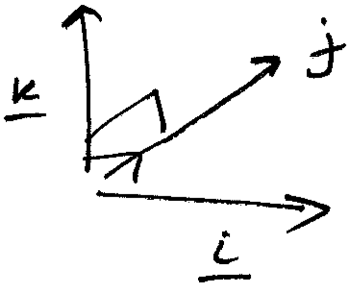
Lecture 24

Previously: vectors geometrically

Today: vectors algebraically

(<u>same</u> vectors, <u>different</u> perspective)

Choose unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ where $\underline{k} = \underline{i} \times \underline{j}$



We introduce coordinates. Each vector $\underline{a}$ can be written $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$ where a_1, a_2, a_3 are called the <u>components</u> of $\underline{a}$.

- $\underline{0} = 0\underline{i} + 0\underline{j} + 0\underline{k}$
- add vectors by adding components

$$\underline{a} + \underline{b} = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) + (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$$
$$= (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} + (a_3 + b_3)\underline{k}$$

$$\lambda \underline{a} = \lambda a_1 \underline{i} + \lambda a_2 \underline{j} + \lambda a_3 \underline{k}$$

We now obtain the coordinate form of the <u>inner product</u> and the <u>vector product</u>.

<u>Inner products</u> Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$,

and $\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$.

Then $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

<u>Proof</u>

$\cdot$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	1	0	0
$\underline{j}$	0	1	0
$\underline{k}$	0	0	1

We use the fact that $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

$$\underline{a} \cdot \underline{b} = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \cdot \underline{b}$$
$$= (a_1 \underline{i}) \cdot \underline{b} + (a_2 \underline{j}) \cdot \underline{b} + (a_3 \underline{k}) \cdot \underline{b}$$
$$= a_1 (\underline{i} \cdot \underline{b}) + a_2 (\underline{j} \cdot \underline{b}) + a_3 (\underline{k} \cdot \underline{b})$$

$$\underline{i} \cdot \underline{b} = \underline{i} \cdot (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$$
$$= \underline{i} \cdot (b_1 \underline{i}) + \underline{i} \cdot (b_2 \underline{j}) + \underline{i} \cdot (b_3 \underline{k})$$

$$= b_1 (\underline{i} \cdot \underline{i}) + b_2 (\underline{i} \cdot \underline{j}) + b_3 (\underline{i} \cdot \underline{k})$$

$$= \underline{\underline{b_1}}$$

Simly.

$$\underline{j} \cdot \underline{b} = b_2$$

$$\underline{k} \cdot \underline{b} = b_3$$

The result now follows ∎

Observe $\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2$

$$\therefore \quad \| \underline{a} \| = +\sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\underline{a} \cdot \underline{b} = \|\underline{a}\| \, \|\underline{b}\| \cos\theta$$

$$\therefore \quad \cos\theta = \frac{\underline{a} \cdot \underline{b}}{\|\underline{a}\| \, \|\underline{b}\|}$$

$$\boxed{\therefore \quad \cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \, \sqrt{b_1^2 + b_2^2 + b_3^2}}}$$

We can therefore easily compute the angle between vectors in coordinate form using the inner product.

Vectors products

Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$

$\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$

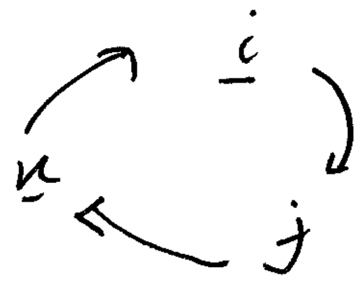
Then $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \underline{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_2 \end{vmatrix} - \underline{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$

Proof

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	
$\underline{j}$		$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$		$\underline{0}$



Now use $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$ for the rest

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$-\underline{k}$	$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$	$-\underline{i}$	$\underline{0}$

We use the fact that $\underline{a}\times(\underline{b}+\underline{c})$

$= \underline{a}\times\underline{b}+\underline{a}\times\underline{c}$

$$\underline{a} \times \underline{b} = \underline{a} \times (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k})$$

$$= \underline{a} \times (a_1 \underline{i}) + \underline{a} \times (a_2 \underline{j}) + \underline{a} \times (a_3 \underline{k})$$

$$= a_1 (\underline{a} \times \underline{i}) + a_2 (\underline{a} \times \underline{j}) + a_3 (\underline{a} \times \underline{k})$$

$$\underline{a} \times \underline{i} = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \times \underline{i}$$

$$= (a_1 \underline{i}) \times \underline{i} + (a_2 \underline{j}) \times \underline{i} + (a_3 \underline{k}) \times \underline{i}$$

$$= \cancel{a_1 (\underline{i} \times \underline{i})} + a_2 (\underline{j} \times \underline{i}) + a_3 (\underline{k} \times \underline{i})$$

$$= -a_2 \underline{k} + a_3 \underline{j}$$

Now carry out similar calculations for

$\underline{a} \times \underline{j}$ and $\underline{a} \times \underline{k}$ to get the result. ∎

<u>Geometric meaning of dets</u>

We define a new product of vectors called the <u>scalar triple product</u>

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c})$$

Using our results for the inner product and the vector product, it is easy to show that

$$[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The inner products and vector products can be considered to get ~~the~~ determinants. We shall use this result in the next lecture. |