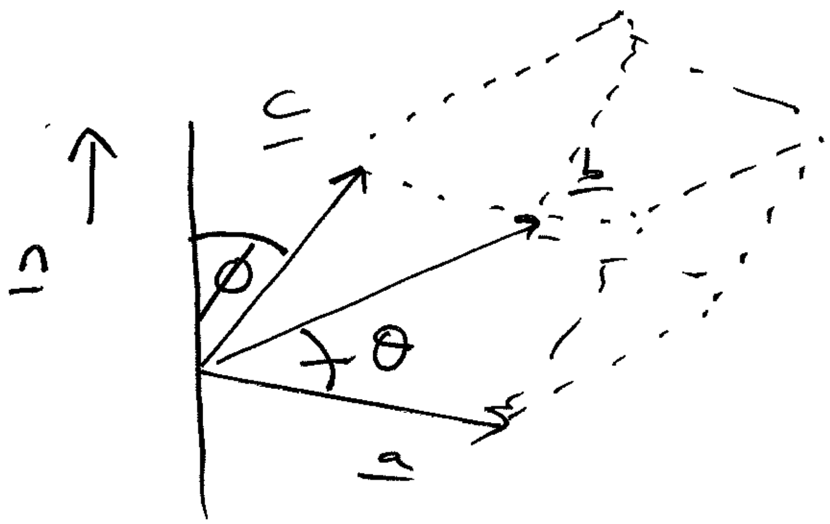
Lecture 25

Theorem Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$
$\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$
$\underline{c} = c_1 \underline{i} + c_2 \underline{j} + c_3 \underline{k}$.

Then the volume of the parallelepiped determined by $\underline{a}$, $\underline{b}$ and $\underline{c}$ is the absolute value of the det.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof



Volume = absolute value of (base area × vertical height)

$$= \left| \, \|\underline{a}\| \, \|\underline{b}\| \cos\theta \cdot \|\underline{c}\| \sin\phi \, \right|$$

Observe that

$$\underline{a} \times \underline{b} = \|\underline{a}\| \, \|\underline{b}\| \sin\theta \, \underline{\hat{n}}$$

$$\underline{c} \cdot (\underline{a} \times \underline{b}) = \|\underline{c}\| \, \|\underline{a}\| \, \|\underline{b}\| \sin\theta \cos\phi$$

$$\therefore \text{ Volume of parallelepiped} = \left| \underline{c} \cdot (\underline{a} \times \underline{b}) \right|$$

$$= \left| [\underline{c}, \underline{a}, \underline{b}] \right|$$

We now use the fact that if you interchange two rows of a det. the det changes by -1.

So

$$[\underline{c}, \underline{a}, \underline{b}] = -[\underline{a}, \underline{c}, \underline{b}]$$

$$= [\underline{a}, \underline{b}, \underline{c}]$$

■

Examples Let

$\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$

$\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$

$\underline{c} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

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1. Calculate $\|\underline{a}\|$.

$$\|\underline{a}\| = \sqrt{\underline{a}\cdot\underline{a}} = \sqrt{1+4+9} = \sqrt{14}$$

2. Calculate $\|\underline{b}\|$

$$\|\underline{b}\| = \sqrt{\underline{b}\cdot\underline{b}} = \sqrt{1+1+4} = \sqrt{6}$$

3. Calculate $\underline{a}\cdot\underline{b}$.

$$\underline{a}\cdot\underline{b} = (\underline{i} + 2\underline{j} + 3\underline{k})\cdot(\underline{i} - \underline{j} + 2\underline{k})$$

$$= 1 - 2 + 6 = \underline{\underline{5}}$$

4. Calculate the angle between $\underline{a}$ and $\underline{b}$ to the nearest degree.

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{\|\underline{a}\|\,\|\underline{b}\|} = \frac{5}{\sqrt{14}\sqrt{6}} \approx 0.546$$

$$\therefore\ \theta \approx 57^{\circ}$$

5. Calculate $\underline{a}\times\underline{b}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}\underline{i} - \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\underline{j} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}\underline{k}$$

$$= 7\underline{i} + \underline{j} - 3\underline{k}$$

Check $(\underline{i} + 2\underline{i} + 3\underline{k})\cdot(7\underline{i} + \underline{j} - 3\underline{k})$

$= 7 + 2 - 9 = 0$ ✓

$(\underline{i} - \underline{j} + 2\underline{k})\cdot(7\underline{i} + \underline{j} - 3\underline{k}) = 7 - 1 - 6 = 0$ ✓

6. Calculate $[\underline{c}, \underline{a}, \underline{b}]$.

$$[\underline{c}, \underline{a}, \underline{b}] = \underline{c} \cdot (\underline{a} \times \underline{b})$$

$$= (2\underline{i} + 3\underline{j} + 4\underline{k}) \cdot (7\underline{i} + \underline{j} - 3\underline{k})$$

$$= 5$$

7. Calculate $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$

$$= 2\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(4+3) - 3(2-3) + 4(-1-2)$$

$$= 14 + 3 - 12 = 5$$

[The answers (6) & (7) should, of course, be the same].

Important

Position vectors
(= bound vectors)

The vectors we have studied so far are often called free vectors.

Now choose a fixed point O called the origin. A position vector is a vector rooted at O. Position vectors enable us to describe specific points — that is, give their coordinates in vector form.

The general position vector is denoted by

$\underline{r} \ (= x\underline{i} + y\underline{j} + z\underline{k})$.

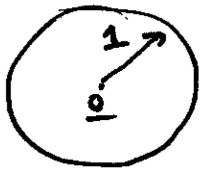
Equations of lines and planes

Curves can be described in two ways

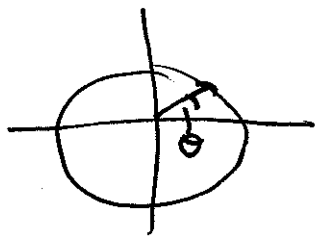
— <u>parametric</u> or <u>non-parametric</u>

[parametric = uses a parameter]

<u>Example</u>



the circle



<u>non-parametric</u>

$x^2 + y^2 = 1$

<u>parametric</u>

$(\cos\theta, \sin\theta)$

$0 \leq \theta \leq 2\pi$

θ is the parameter.

Cartesian coordinates = those given by x, y, z

↕

(Descartes)