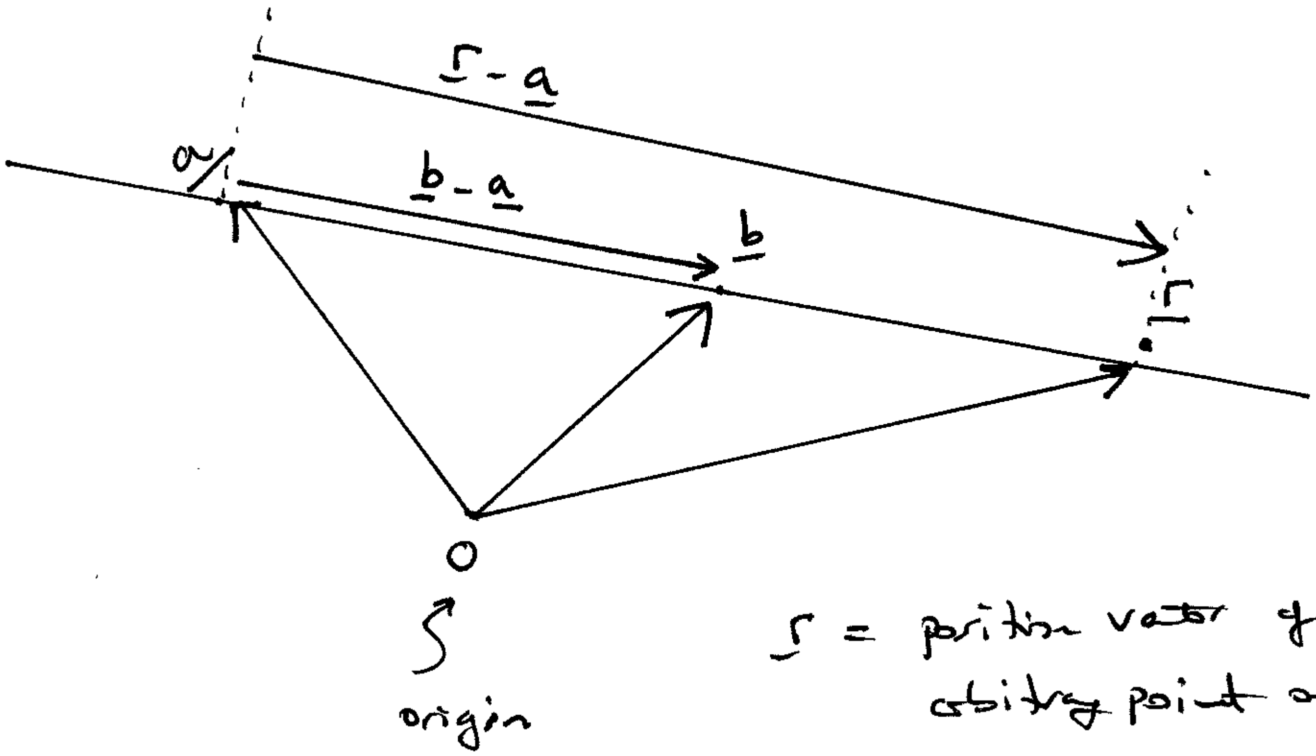
Lecture 26

Equations of the line

<u>Given</u>: the position vectors $\underline{a}$ and $\underline{b}$ of two distinct points

<u>Find</u>: the equation of the (unique) line that contains these two points.



$\underline{r}$ = position vector of arbitrary point on the line.

$\underline{b} - \underline{a} \neq \underline{0}$ and is the direction in space of the line (+ve and −ve).

$\underline{r} - \underline{a}$ is parallel to $\underline{b} - \underline{a}$

$\therefore \quad \underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a})$ for some $\lambda \in \mathbb{R}$

parameter

$$\boxed{\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})}$$

the parametric vector equation of the line

Observe when $\lambda = 0$, $\underline{r} = \underline{a}$

when $\lambda = 1$, $\underline{r} = \underline{b}$

So this line does contain the original two points.

($\underline{b} \neq \underline{0}$)

We say that vectors $\underline{a}$ and $\underline{b}$ are parallel if $\underline{a} = \lambda \underline{b}$ for some $\lambda \in \mathbb{R}$

Observe also that $\underline{c} = \underline{b} - \underline{a}$ is a direction that the line is parallel to.

It follows that a line in space is also determined by a position vector (that gives a point on the line) and a (free) vector that is parallel to the the line.

$$\underline{r} = \underline{a} + \lambda \underline{c}$$

Write in component form

$$x\underline{i} + y\underline{j} + z\underline{k} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) + \lambda(c_1\underline{i} + c_2\underline{j} + c_3\underline{k})$$

$$\text{RHS} = (a_1 + \lambda c_1)\underline{i} + (a_2 + \lambda c_2)\underline{j} + (a_3 + \lambda c_3)\underline{k}.$$

Now equate components:

$$\boxed{\begin{aligned} x &= a_1 + \lambda c_1, \\ y &= a_2 + \lambda c_2 \\ z &= a_3 + \lambda c_3 \end{aligned}}$$

parametric Cartesian equations of the line

We now eliminate the parameter.

$$x - a_1 = \lambda c_1 \quad \therefore \quad \text{if } c_1 \neq 0$$

$$\frac{x - a_1}{c_1} = \lambda \quad \text{similarly}$$

$$\frac{y - a_2}{c_2} = \lambda \qquad \text{if } c_2 \neq 0$$

$$\frac{z - a_3}{c_3} = \lambda \qquad \text{if } c_3 \neq 0.$$

It follows that we get two equations

$$\boxed{\begin{gathered} \frac{x-a_1}{c_1} = \frac{y-a_2}{c_2} \\ \\ \frac{y-a_2}{c_2} = \frac{z-a_3}{c_3} \\ \\ \text{Non-parametric Cartesian equation of the line} \end{gathered}}$$

← What these really mean will be explained later.

Example Find the vector parametric equation, the Cartesian parametric equation and the non-parametric Cartesian equation of the line determined by the points with position vectors

$\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{b} = 4\underline{i} + \underline{k}$

Vector parametric equation

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$$

So,

$$\boxed{\underline{r} = \underline{i} + 2\underline{j} + 3\underline{k} + \lambda(3\underline{i} - 2\underline{j} - 2\underline{k})}$$

parametric vector equation.

Cartesian parametric equation

$$x\underline{i} + y\underline{j} + z\underline{k} = (1+3\lambda)\,\underline{i} + (2-2\lambda)\,\underline{j} + (3-2\lambda)\,\underline{k}$$

Now equate components

$$\boxed{x = 1+3\lambda, \quad y = 2-2\lambda, \quad z = 3-2\lambda}$$

parametric Cartesian equations

Non-parametric Cartesian equations

$$\frac{x-1}{3} = \lambda, \quad \frac{y-2}{-2} = \lambda, \quad \frac{z-3}{-2} = \lambda$$

$\therefore$ Non-parametric Cartesian equations

$$\frac{x-1}{3} = \frac{y-2}{-2}$$

$$\frac{y-2}{-2} = \frac{z-3}{-2}$$

Tidy up these equations to get

$$\boxed{\begin{aligned} 2x + 3y &= 8 \\ y - z &= -1 \end{aligned}}$$

Non-parametric Cartesian equations of line. (*)

Two linear equations in the unknowns x, y, z

In coordinate form $\underline{a}$ is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\underline{b}$ is $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

both of these satisfy the equations (*)