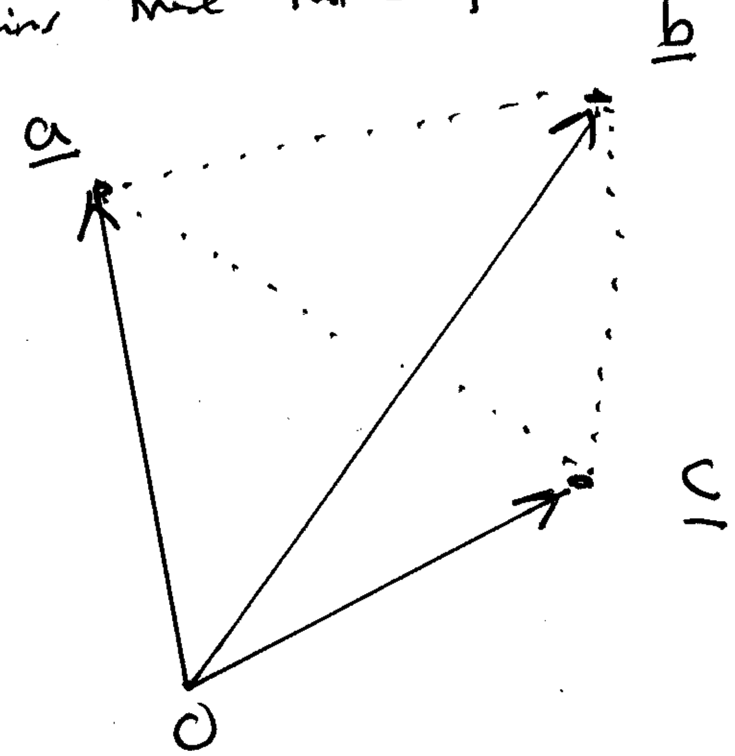
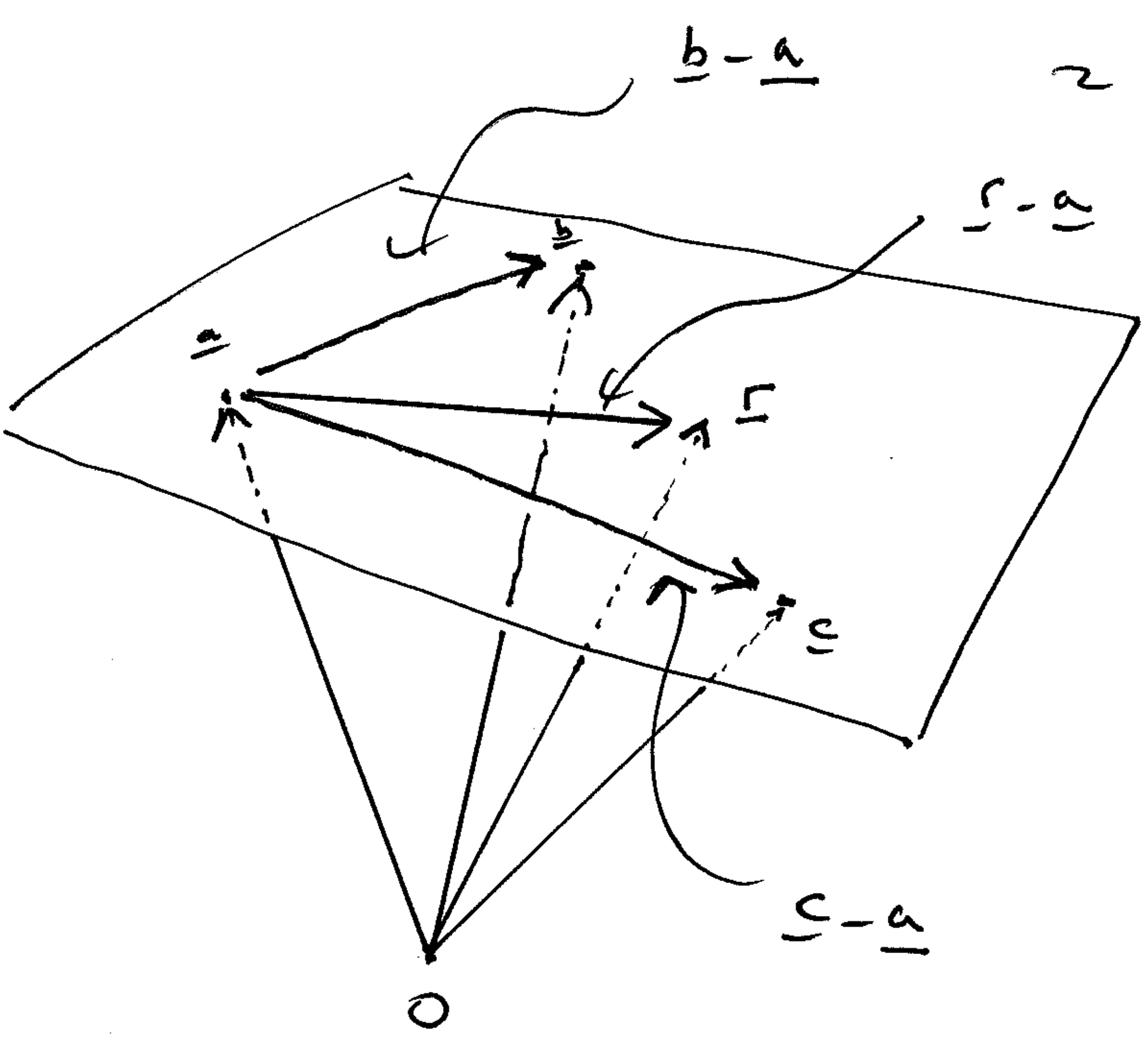
Lecture 27

Equation of the plane

<u>Given:</u> the position vectors of three points that form a triangle $\underline{a}, \underline{b}, \underline{c}$.

<u>Find:</u> the equation of the unique plane that contains these three points.





$\underline{r} - \underline{a}$ is parallel to the plane determined by the vectors $\underline{b} - \underline{a}$ and $\underline{c} - \underline{a}$. It follows that

$$\underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$$

for $\lambda, \mu \in \mathbb{R}$

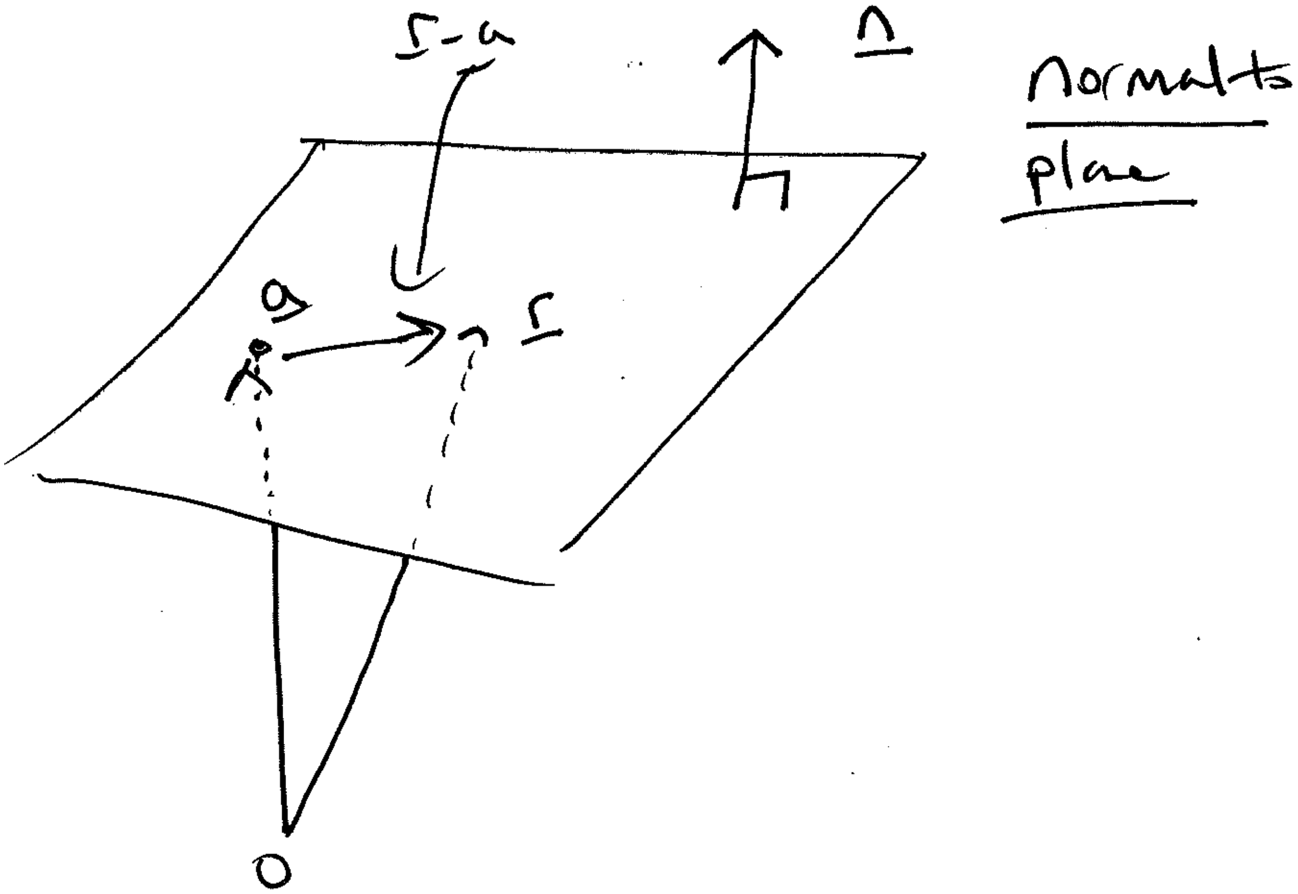
parameters

$$\boxed{\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})}$$

parametric vector equation det'd by $\underline{a}, \underline{b}, \underline{c}$.

Observe that $\underline{b} - \underline{a}$ and $\underline{c} - \underline{a}$ are vectors parallel to the plane determined by $\underline{a}, \underline{b}, \underline{c}$.

To obtain the non-parametric cartesian equation we take a different tack.



Given: position vector of point in plane and a vector normal to the plane (ie orthogonal to it)

Observe that

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0.$$

i.e. $$\boxed{\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}}$$

We can now write down the non-parametric Cartesian equation of the plane

$$n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$$

How to find a normal to a plane

Suppose the plane is def'd by the position vectors of three points $\underline{a}$, $\underline{b}$, $\underline{c}$. Then

$$\underline{n} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a})$$

is a normal.

Systems of linear equations in 3 unknowns revisited.

The equation

$$ax + by + cz = d$$

represents, in general, a plane in space

Exceptions:

- $a = b = c = d = 0$ which is all of $\mathbb{R}^3$
- $a = b = c = 0$, $d \neq 0$ which is $\emptyset$

In general, the intersection of two sub planes will be a line — this accounts for the non-parametric equation of the line we derived in the last lecture.

In general, the intersection of three sub planes will be a point.

But in both the above cases, there are exceptions if the planes are angled in particular ways.

<u>Example</u> Determine the parametric equation of the non-parametric equation determined by the three points with position vectors

$$\underline{a} = \underline{i} - \underline{j} + 3\underline{k}$$

$$\underline{b} = 4\underline{i} + \underline{j} - 2\underline{k}$$

$$\underline{c} = -\underline{i} - \underline{j} + \underline{k}$$

The parametric equation of the plane is given by

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$$

Thus

$$\underline{r} = (\underline{i} - \underline{j} + 3\underline{k}) + \lambda(3\underline{i} + 2\underline{j} - 5\underline{k})$$

$$+ \mu(-2\underline{i} + 0\underline{j} \cancel{+} -2\underline{k})$$

<u>Check</u> when $\lambda = 0, \mu = 0$ $\underline{r} = \underline{a}$ ✓

when $\lambda = 1, \mu = 0$, $\underline{r} = \underline{b}$ ✓

when $\lambda = 0$ $\mu = 1$, $\underline{r} = \underline{c}$ ✓

The non-parametric Cartesian equation

$$\underline{n} = (\underline{b}-\underline{a}) \times (\underline{c}-\underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -5 \\ -2 & 0 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -5 \\ 0 & -2 \end{vmatrix} \underline{i} - \begin{vmatrix} 3 & -5 \\ -2 & -2 \end{vmatrix} \underline{j} + \begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} \underline{k}$$

$-6-10=+16$

$$= -4\underline{i} + 16\underline{j} + 4\underline{k}$$

Check

$(-4\underline{i} + 16\underline{j} + 4\underline{k}) \cdot (3\underline{i} + 2\underline{j} - 5\underline{k})$

$= -12 + 32 - 20 = 0$ ✓

$(-4\underline{i} + 16\underline{j} + 4\underline{k}) \cdot (-2\underline{i} - 2\underline{k})$

$= 8 - 8 = 0$ ✓

The required equation is

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0 \quad \text{i.e}$$

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

$$-4x + 16y + 4z = -8$$

$$\left[\underline{a} \cdot \underline{n} = (\underline{i} - \underline{j} + 3\underline{k}) \cdot (-4\underline{i} + 16\underline{j} + 4\underline{k}) \right.$$

$$\left. = -4 - 16 + 12 = \underline{\underline{-8}} \right]$$

$$\therefore \quad \boxed{x - 4y - z = 2}$$

<u>Check</u>

$\underline{a}: \quad 1 + 4 - 3 = 2$ ✓

$\underline{b}: \quad 4 - 4 + 2 = 2$ ✓

$\underline{c}: \quad -1 + 4 - 1 = 2$ ✓