Lecture 6

We have met the notion of an (ordered) k-tuple $(x_1, \dots, x_k)$. In what follows, $x_1, \dots, x_n \in X$ some fixed set with n elements. In general, we allow repetitions in k-tuples.

Definition If $(x_1, \dots, x_k)$ is a k-tuple in which all $x_1, \dots, x_k$ are different is called a **k-permutation** over X.

n-perm = permutation

Example Let $X = \{a, b, c, d\}$.

We find all 2-perms over X.

~~(a,b) (b,a) (a,c) (b,c) (a,d) $($~~

$(a,b), (a,c), (a,d)$

$(b,a), (b,c), (b,d)$

$(c,a), (c,b), (c,d)$

$(d,a), (d,b), (d,c)$

12 2-perms

<u>Theorem 2</u> Let X be a set with $|X| = n$.

Let $0 \le k \le n$. The number of k-perms over X is

$$\boxed{{}^nP_k = \frac{n!}{(n-k)!}}$$

<u>Example</u> $4P_2 = \frac{4!}{2!} = \underline{\underline{12}}$

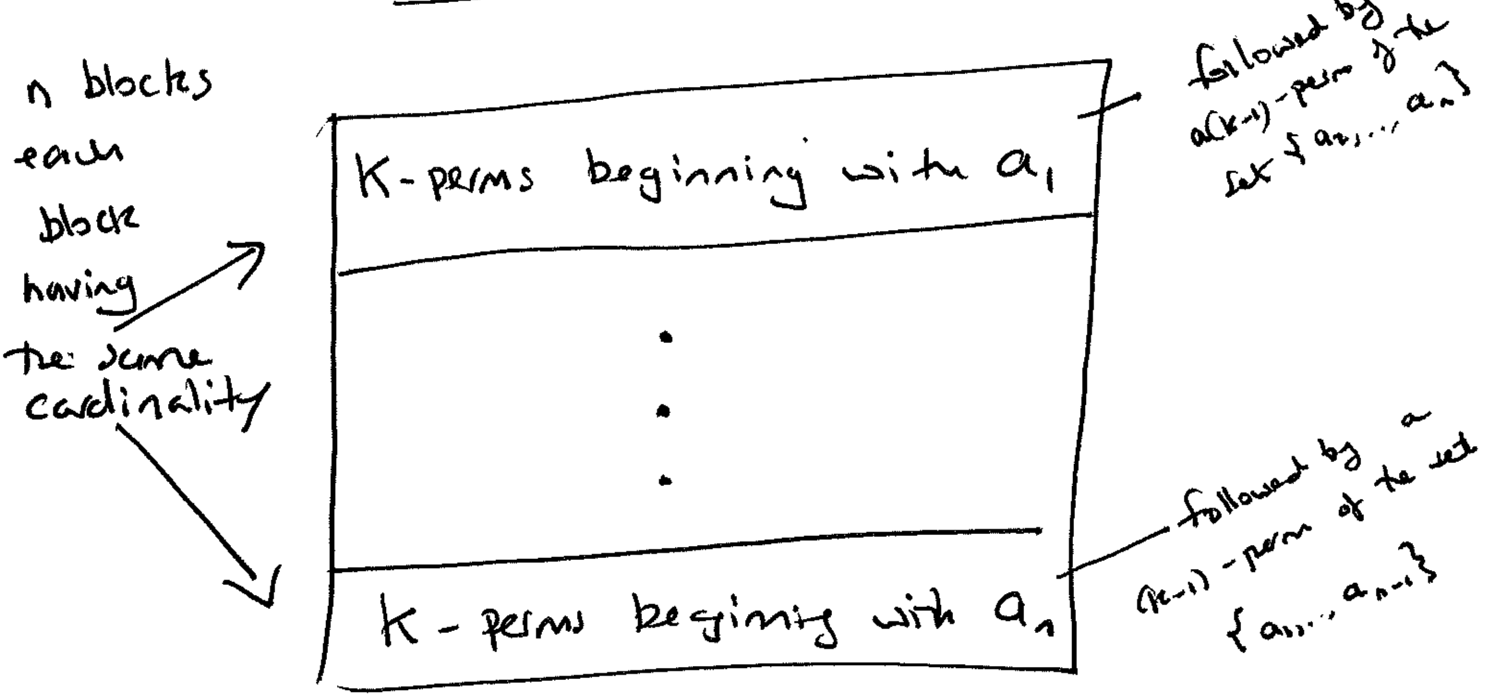
Observe that ${}^nP_0 = \underline{1}$

all perms $\longrightarrow$ ${}^nP_n = n!$ (explain why $0! = 1$)

Proof Let the set $X = \{a_1, \dots, a_n\}$.

The set of all k-perms can be partitioned

All k-perms of X



$$^nP_k = n\ {}^{n-1}P_{k-1}$$

Now repeat ...

$$
\begin{aligned}
{}^{n}P_{k} &= n\; {}^{n-1}P_{k-1} \\
&= n\left[(n-1)\; {}^{n-2}P_{k-2} \right] \\
&= n(n-1)(n-2)\; {}^{n-3}P_{k-3} \\
&\;\; \cdots\cdots\cdots \\
&= n(n-1)\cdots(n-(k-1))\; \underbrace{{}^{n-k}P_{0}}_{=\,1} \\
&= \frac{n!}{(n-k)!}
\end{aligned}
$$

Theorem 3 Let X be a set with $|X| = n$. Then the number of k-subsets is

$$\underbrace{{}^{n}C_{k} = \binom{n}{k}}_{\substack{\text{different notation} \\ \text{same meaning}}} = \frac{n!}{k!\,(n-k)!}$$

called a **binomial coefficient**.

read 'n choose k'

Proof The number of k-perms is $\frac{n!}{k!\,(n-k)!}$. Partition the set of all such k-perms so that the elements of a block are k-perms of the same subset of X.

Each block contains $k!$ elements
(the number of perms of k elements).

The number of blocks in the partition is therefore $\frac{1}{k!} \frac{n!}{(n-k)!}$.

But there is a bijection between the set of blocks and the set of k-subsets. ☒

Example follows...

Example

Let $X = \{a, b, c, d\}$

All 2-perms over X	2-subsets over X
$(a,b), (b,a)$	$\{a,b\}$
$(a,c), (c,a)$	$\{a,c\}$
$(a,d), (d,a)$	$\{a,d\}$
$(b,d), (d,b)$	$\{b,d\}$
$(b,c), (c,b)$	$\{b,c\}$
$(c,d), (d,c)$	$\{c,d\}$

We partition the set of k-perms

This illustrates why

$$k!\,{}^nC_k = {}^nP_k$$

<u>Example</u> In the National lottery, you have to choose 6 numbers from 59. Thus the number of possible ways of doing this is $\binom{59}{6}$

$= 45,057,474.$

See <u>Quiz 1</u> for more thought-provoking exercises that fill in some blanks (such as our observation on Pascal's Δ)

END

OF

SECTION

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