Lecture 7

2. Polynomials

- study roots of polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

i.e. find all numbers r s.t. $p(r) = 0$

- introduce the complex numbers.

Section 4·3 : Solving quadratic equations

Probably mainly revision.
But I will introduce some important terminology.

Let a, b, c be real numbers where $a \neq 0$.

Then $ax^2 + bx + c$ is called a quadratic polynomial / polynomial of degree 2

The equation

$$\underbrace{a}_{\text{leading coefficient}}x^2 + bx + c = 0 \quad (*)$$

is called a quadratic equation

If r is a real number (for the time being) such that

$$ar^2 + br + c = 0$$

then it is called a root of the quadratic equation.

There is a well-known formula for finding the roots of a quadratic equation.

We now derive that equation.

$$ax^2+bx+c=0 \qquad a\neq 0$$

Divide through by a (since $a\neq 0$!)

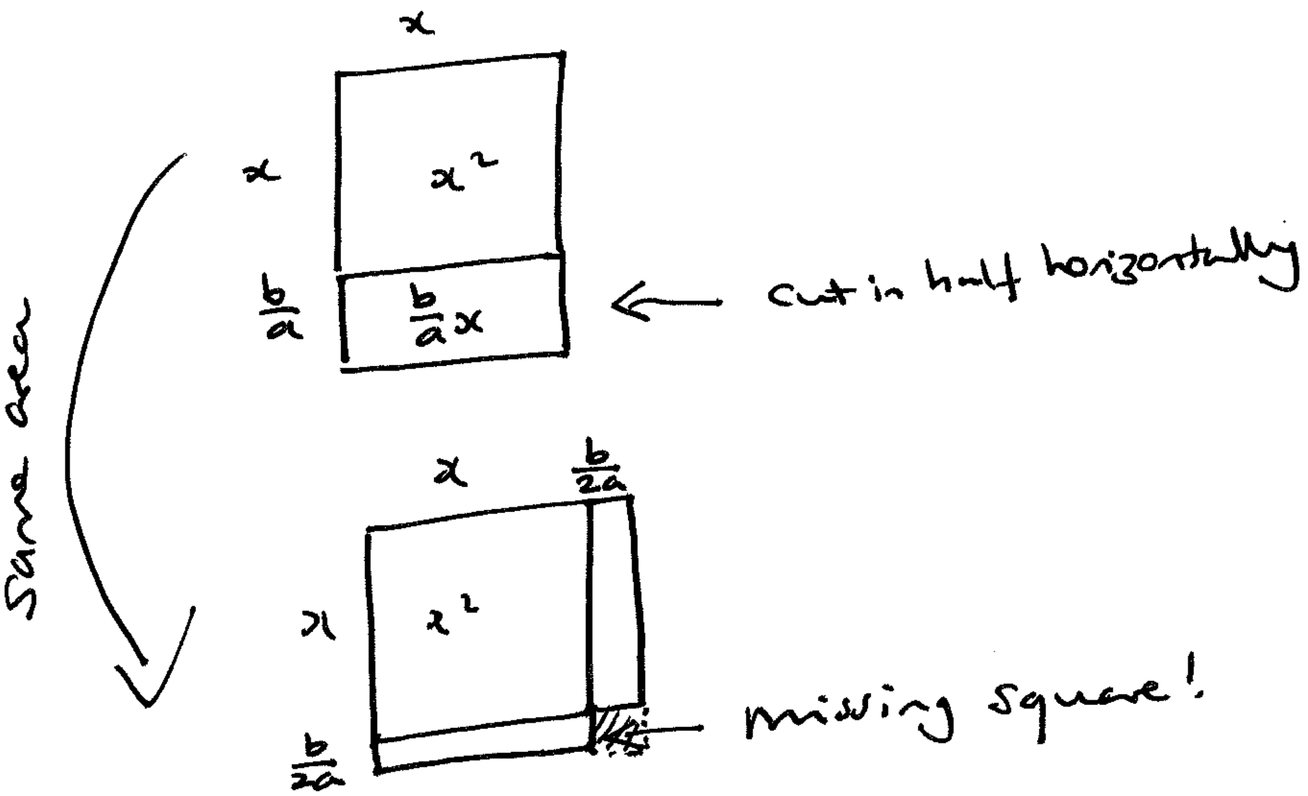
monic since leading coefficient is equal to 1

$$\boxed{x^2+\frac{b}{a}x}+\frac{c}{a}=0.$$

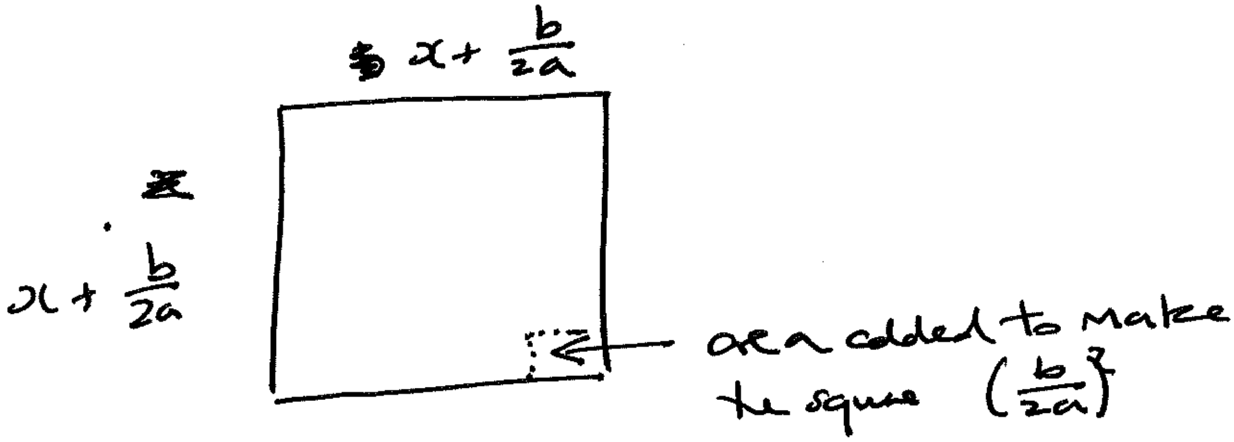
constant

focus on this.

We draw a picture of $x^2+\frac{b}{a}x$.



Complete the square



Comparing pictures (and checking ~~to~~ by algebra)

$$\boxed{x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2}$$

The pictures explain the name of this ~~equation~~ identity: <u>Completing the square</u>.

The quadratic equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

is therefore equal to

$$\left(x + \frac{b}{2a}\right)^2 - \underbrace{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}}_{\text{constant.}} = 0$$

This can be written as

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$= \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= \frac{\pm 1}{2a}\sqrt{b^2-4ac}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

This is the formula you know and love from school.

Define $D = b^2 - 4ac$

called the <u>discriminant</u> of ~~f~~ our quadratic polynomial.

There are now three cases depending on the sign of D

(1) If $D > 0$ then our quadratic equation has two, distinct real roots.

(2) If $D = 0$ then $-\frac{b}{2a}$ is one repeated root — still two roots but the same root is counted twice (explained later)

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

$$= x^2 + \frac{b}{a}x + \frac{\cancel{4}ac}{\cancel{4}a^2}$$

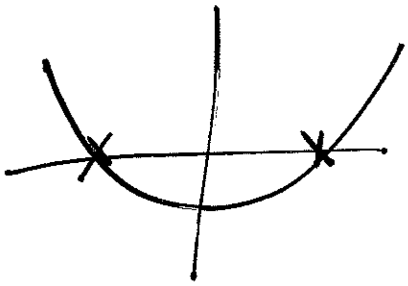
$$= x^2 + \frac{b}{a}x + \frac{c}{a}.$$

(3) If $D<0$ there are no real roots

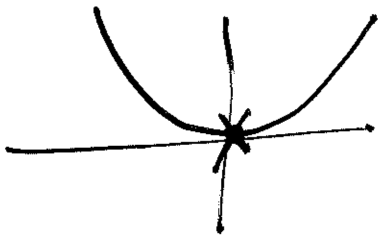
We say that our quadratic polynomial is irreducible.

Pictures

$D>0$



$D=0$



$D<0$