Lecture 8

Axioms for highschool algebra

(Think of the real numbers $\mathbb{R}$)

Addition

(F1) Addition is associative

$$(x+y)+z = x+(y+z)$$

(F2) There is an additive identity 0:

$$x+0 = x = 0+x.$$

(F3) Each number has an additive inverse.

For each x there is a (unique) $-x$ such that

$$x+(-x) = 0 = (-x)+x.$$

(F4) Addition is commutative.

$$x+y = y+x.$$

Multiplication

(F5) Multiplication is associative.

$$x(yz) = (xy)z.$$

(F6) There is a multiplicative identity 1:

$$1x = x = x1.$$

(F7) Each non-zero number has a multiplicative inverse. That is: for each $x \neq 0$ there exists a number x^{-1} s.t. $x x^{-1} = 1 = x^{-1} x$.

(F8) Multiplication is associative.

Linking axioms
(connecting addition and multiplication)

(F9) $0 \neq 1$.

(F10) The additive identity is a multiplicative zero. That is $0x = 0 = x0$.

(F11) Multiplication distributes over addition.

$$x(y+z) = xy + xz$$

$$(y+z)x = yx + zx.$$

For later reference Any set with operations satisfying these operations is called a field

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (later) are all fields.

These properties are the hidden properties of highschool algebra — the rules of the game.

Definitions

subtraction $\quad a - b \overset{\text{def}}{=} a + (-b)$

Division $\quad \dfrac{a}{b} = a/b \overset{\text{def}}{=} a \times b^{-1}$

(assuming $b \neq 0$)

In a field, addition, subtraction, multiplication and division all have the usual properties.

Example Solve $ax = b$ when $a \neq 0$.

$$ax = b$$

$$\bar{a}^1(ax) = \bar{a}^1 b \quad \text{by (F7)}$$

$$(\bar{a}^1 a)x = \bar{a}^1 b \quad \text{by (F5)}$$

$$1x = \bar{a}^1 b \quad \text{by (F7)}$$

$$\underline{\underline{x = \bar{a}^1 b}} \quad \text{by (F6)}$$

<u>Example</u> $-1 \times -1 = 1$

[proved using only the field axioms]

We show that $(-1)(-1) - 1 = 0$.

$$
\begin{aligned}
(-1)(-1) - 1 &= (-1)(-1) + (-1) && \text{definition} \\
&= (-1)(-1) + (-1)\,1 && \text{by (F6)} \\
&= (-1)\left[(-1) + 1\right] && \text{by (F11)} \\
&= (-1)\,0 && \text{by (F3)} \\
&= 0 && \text{by (F10)}
\end{aligned}
$$

The field axioms enable you to check results in algebra but they do not on their own lead to these results. This requires creativity