Lecture 9

Notation Because of the associativity of addition, we can write a sum such as

$$x_1 + \dots + x_m$$

without using brackets. We usually abbreviate this as

$$\sum_{i=1}^{m} x_i$$

Pay attention to the limits of the summation

$$\sum_{i=0}^{10} x_i = x_0 + x_1 + \dots + x_{10}$$

We shall now consider ~~s~~ one of our counting results from Section 1 with some algebra.

The expression $a+b$ (or $x+y$, ...) is called a <u>binomial</u> (it is in two parts).

The <u>binomial theorem</u> tells us how to <u>write down</u> the expansion of $(x+y)^n$ without having to work it out each time.

To understand how to do this, we look first at a special ~~example~~ case.

We shall use the field axioms.

<u>Example</u> We calculate $(x+y)^3$ in slow motion to see what is happening.

$$(x+y)^3 \overset{def}{=} \underbrace{(x+y)(x+y)(x+y)}$$

We expand this by repeated use of the distributive law.
I assume associativity throughout.

$$\begin{aligned}(x+y)(x+y) &= (x+y)x + (x+y)y \\ &= xx + yx + xy + yy\end{aligned}$$

$$[(x+y)(x+y)](x+y)$$

$$= (xx + yx + xy + yy)(x+y)$$

$$= (xx + yx + xy + yy)x + (xx + yx + xy + yy)y$$

$$= \underset{①}{xxx} + \underset{②}{yxx} + \underset{③}{xyx} + \underset{④}{yyx} + \underset{⑤}{xxy} + \underset{⑥}{yxy} + \underset{⑦}{xyy} + \underset{⑧}{yyy}$$

So far, I have just used <u>associativity</u> & distributivity. We obtain a sum with 8 terms.

Lets group the terms according to whether they are equal or not (using commutativity)

xxx	$=$	$1x^3$
yxx xyx xxy	$=$	$3x^2y$
yyx yxy xyy	$=$	$3xy^2$
yyy	$=$	$1y^3$

Claim The number of sequences (n-tuple) of x's and y's with exactly i x's is $\binom{n}{i}$

[The set of all sequences is in bijective correspondence with all i-subsets of an n-element set]

proves

Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof $(x+y)^n = \overbrace{\underbrace{x \dots x}_{n \text{ elements}} + \dots\dots + \underbrace{y \dots y}_{n \text{ elements}}}^{\text{all possible sequences of } x\text{s and } y\text{s of length } n}$

The # of terms of xs and ys of length n with exactly i xs is $\binom{n}{i}$. And such terms equal $x^i y^{n-i}$ by commutativity and associativity. ∎

<u>Example</u> Expand $(2a-3b)^6$ using the binomial theorem.

needs to be in the form $(x+y)^6$

$$(2a-3b)^6 = (2a+(-3b))^6$$

$$= \sum_{i=0}^{6} \binom{6}{i} (2a)^i (-3b)^{6-i}$$

$$= \sum_{i=0}^{6} \binom{6}{i} 2^i a^i b^{6-i} (-1)^{6-i} 3^{6-i}$$

$$= \sum_{i=0}^{6} \boxed{\binom{6}{i} (-1)^{6-i} 2^i 3^{6-i}} a^i b^{6-i}$$

This number is called the <u>coefficient</u> of $a^i b^{6-i}$

A <u>coefficient</u> is always a number

Calculate the coefficient of a^2b^4 in this expression.

<u>Put $i = 2$</u>

Coefficient of a^2b^4 = $\binom{6}{2}(-1)^4\, 2^2\, 3^4$

$= \underline{\underline{4{,}860}}$

$$\boxed{\text{Careful!} \quad (st)^n = s^n t^n}$$

Aside on exponents

Laws of exponents $a^{m+n} = a^m a^n$, $(a^m)^n = a^{mn}$. (where $m, n \geq 1$). We want these laws to persist for more general exponents.

$$a^m \cdot a^{-m} = a^{m-m} = a^0$$

But $a^m \cdot a^{-m} = \dfrac{a^m}{a^m} = 1$

$$\therefore \boxed{a^0 = 1}$$

What is a^b when $a, b \in \mathbb{R}$? [$\sqrt{2}^{\pi}$??]

Heuristic argument

$$\ln(a^b) = b \ln a.$$

But now we are using special functions

$$\therefore \boxed{a^b = \exp(b \ln a)}$$

If $b = 0$ then $a^0 = \exp(0) = 1$

(what happens if $a = 0$? (or $a \to 0$?))