Lecture 4

Functions

Let A and B be sets. A rule that associates with each element of A one element of B is called a function from A to B.

We call A the domain and B the codomain.

If the function is called f we might write

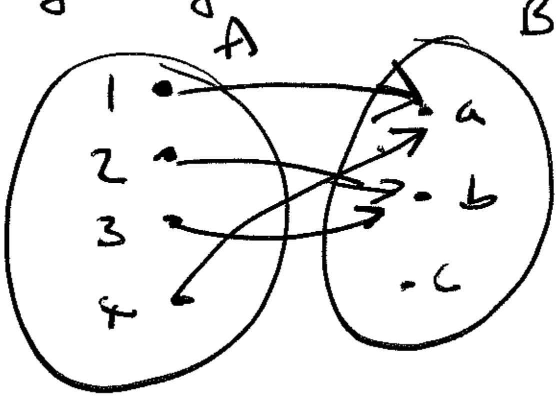
$f : A \to B$ or $A \xrightarrow{f} B$.

If $a \in A$, then the element f associates with a in B is written $f(a)$. We often write

$a \mapsto f(a)$. The image of f

$= \{ f(a) : a \in A \} = \mathrm{im}(f)$.

<u>Example</u> Simple examples of functions can be illustrated by means of diagrams called <u>arrow diagrams</u>. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. The function f is defined by the following arrow diagram



or, equivalently, by means of the following table

A	B
1	a
2	b
3	b
4	a

Special kinds of functions

Let $f : A \to B$ be a function. We say that f is injective if $a, a' \in A$ and $a \neq a'$ implies $f(a) \neq f(b)$. We say that f is surjective if for each $b \in B$, there exists $a \in A$ s.t. $f(a) = b$. A function that is both injective and surjective is called bijective.

Examples

(i) $f : \mathbb{N} \to \mathbb{N}$ where $a \mapsto a+1$ is injective but not surjective since 0 is not an element of the image of f.

(ii) $f : \mathbb{Z} \to \mathbb{N}$ given by

$$f(a) = |a| \quad \leftarrow \text{absolute value of } a$$

is surjective but not injective (since $f(-1) = 1 = f(1)$)

(iii) $f : \{1,2,3\} \to \{a,b,c\}$ given by $f(1) = a$, $f(2) = b$, $f(3) = c$ is bijective.

<u>Definition</u> Let A be a set.

A function $f : A \times A \to A$ is called a <u>binary operation</u> on A

<u>Notation</u> We usually do not write $f(a,b)$ but something like $a * b$.

<u>Examples</u>

(i) $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$(m,n) \mapsto m+n$$ is the

binary operation of addition

$$(m,n) \mapsto m \times n$$

is the binary operation of multiplication.

We now turn to the properties of the Boolean operations $A \cap B$ and $A \cup B$.

> Whenever you meet new operations, always ask what <u>properties</u> those operations have.

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Properties of Boolean operations

This is a version of Section 3.2.

Key definitions

- Let X be a non-empty set. A *binary operation* $*$ on X associates with every ordered pair (x, y) of elements of X a single element $x * y$ of X.
- The binary operation $*$ is said to be *commutative* if $x * y = y * x$ for all $x, y \in X$.
- The binary operation $*$ is said to be *associative* if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.
- An *identity* for $*$ is an element e such that $e * x = x = x * e$ for all $x \in X$.
- A *zero* for $*$ is an element z such that $z * x = z = x * z$ for all $x \in X$.
- The binary operation $*$ is said to be *idempotent* if $x * x = x$ for all $x \in X$.
- If $\bullet$ is another binary operation on the set X we say that $*$ *distributes over* $\bullet$ if $x * (y \bullet z) = (x * y) \bullet (x * z)$.

Properties

Let A, B and C be any sets.

(1) $A \cap (B \cap C) = (A \cap B) \cap C$. Intersection is associative.
(2) $A \cap B = B \cap A$. Intersection is commutative.
(3) $A \cap \emptyset = \emptyset = \emptyset \cap A$. The empty set is the zero for intersection.
(4) $A \cup (B \cup C) = (A \cup B) \cup C$. Union is associative.
(5) $A \cup B = B \cup A$. Union is commutative.
(6) $A \cup \emptyset = A = \emptyset \cup A$. The empty set is the identity for union.
(7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Intersection distributes over union.
(8) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Union distributes over intersection.
~~(9) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. De Morgan's law part one.~~
~~(10) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. De Morgan's law part two.~~
(~~11~~ 9) $A \cap A = A$. Intersection is idempotent.
(~~12~~ 10) $A \cup A = A$. Union is idempotent.