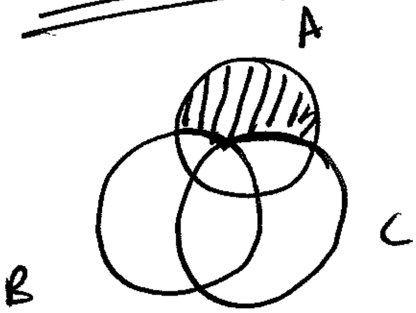
Lecture 5

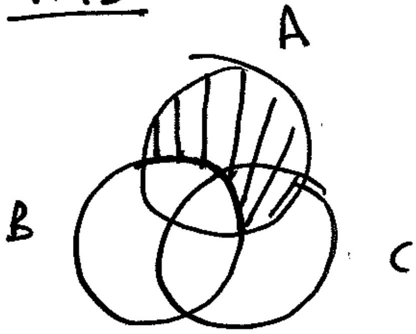
Properties of Boolean operations

We illustrate $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

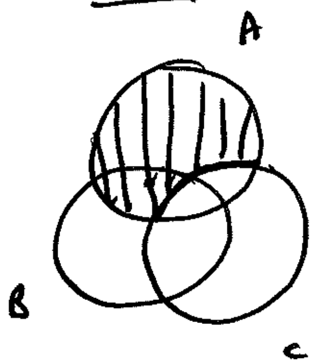
$A \setminus (B \cup C)$



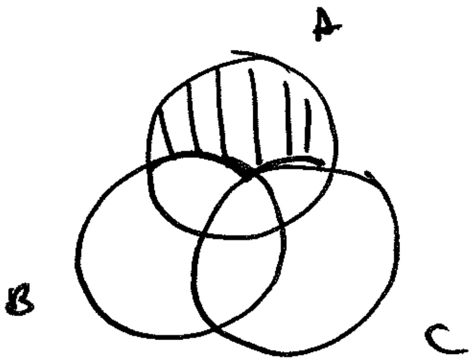
$A \setminus B$



$A \setminus C$



$(A \setminus B) \cap (A \setminus C)$



Proofs of the properties can be found in $\boxed{\text{Section 3.2}}$. We shall return to these properties later on, however.

Important consequences of some of these properties.

Because of associativity, we can write multiple unions and intersections without brackets

$$\left.\begin{array}{l} A_1\cup\dots\cup A_n \\ A_1\cap\dots\cap A_n\end{array}\right\} \text{unambiguous}$$

We would usually write

$$\bigcup_{i=1}^{n} A_i = A_1\cup\dots\cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1\cap\dots\cap A_n$$

I have now concluded describing all the notation we shall need

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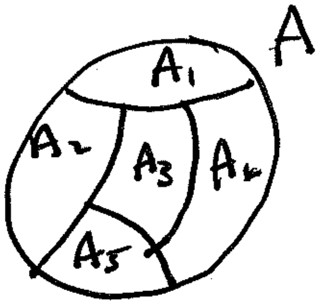
Counting principles

(1) Sets X and Y have the same cardinality ($|X| = |Y|$) precisely when there is a bijection between them.

(2) Let A be a set. Let $A_1, \dots, A_n \subseteq A$ (where $A_i \neq \emptyset$). s.t.



(1) $A = A_1 \cup \dots \cup A_n$.

(2) $A_i \cap A_j = \emptyset$ if $i \neq j$

Then $|A| = |A_1| + \dots + |A_n|$

(3) If $A = A_1 \times \dots \times A_n$. Then $|A| = |A_1| \dots |A_n|$

(1) is really a definition of what it means for two sets to have the same cardinality.

(2) & (3) can be proved, but we shall just assume them here. [See Section 3.9 - but that does more than I cover in this course].

We shall now use these principles to prove some results about counting

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Theorem 1 Let X be any (finite) set. Then $|P(X)| = 2^{|X|}$

We begin by counting a set that is very different from $P(X)$ (the set we want to count).

Let $A = \{0,1\}$. Then $A^n = \underbrace{A \times \ldots \times A}_{n \text{ times}}$

is the set of all binary sequences of length n.

Example

$\{0,1\}^3 = \{ (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1) \}$

$|\{0,1\}^3| = 2^3 = 8.$

By Counting principle (3): $|\{0,1\}^n| = 2^n.$

Let $|X| = n$.

To prove theorem 1, I shall construct a bijection between $\{0,1\}^n$ and $P(\{1,2,..,n\})$

$$f : \{0,1\}^n \longrightarrow P(\{1,2,..,n\})$$

is defined as follows

$f((b_1,..,b_n))$ is the subset of $\{1,2,\dots n\}$ that contains i if $b_i = 1$ and doesn't contain i if $b_i = 0$.

Example $n=3$

binary sequence	subset of $\{1,2,3\}$
$(0,0,0)$	$\emptyset$
$(1,0,0)$	$\{1\}$
$(0,1,0)$	$\{2\}$
$(0,0,1)$	$\{3\}$
$(1,1,0)$	$\{1,2\}$
$(1,0,1)$	$\{1,3\}$

(0, 1, 1)	{2, 3}
(1, 1, 1)	{1, 2, 3}

The function f is a bijection and so

$$|\{0,1\}^n| = |P(\{1,..,n\})|$$

$$\|$$

$$2^n$$

You should think about how you would convince someone that f really was bijective.

Notation for next time

$$0! = 1$$
$$n! = n(n-1)! \quad \text{if } n \geq 1$$

Example of a definition by recursion

Called **n factorial**

Example $3! = 3 \cdot 2 \cdot 1 = \underline{6}$

Let X be a set. (with n elements).

A subset $Y \subseteq X$ with k elements ($0 \leq k \leq n$) is called a **k-subset**

Example Let $X = \{a, b, c, d\}$. We find all 2-subsets of X: $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$.

A k-subset of a set X is sometimes called a <u>combination</u>.

<u>Examples</u> A <u>committee</u> is a set of people. We could ask (though it is pretty boring) for all 4 people committees that could be formed from all students in this lecture. We are therefore looking at 4-subsets of a ≈ 130 element set.

<u>Example</u> The national lottery.

Select/Choose 6 numbers from 1 – 59 (inclusive)

We are therefore interested in 6-subsets of a 59 element set.

[45, 057, 474 ways of doing this]