Lecture 10

Complex numbers Continued.

We have defined a new number i with the property that $\boxed{i^2 = -1}$. All other rules for high-school algebra apply.

<u>Example</u> Calculate the "roots" of $x^2 - 2x + 2$, which is real irreducible.

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = \underline{\underline{1 \pm i}}$$

We now check that $1+i$ and $1-i$ really are roots of x^2-2x+2

- $(1+i)^2 - 2(1+i) + 2$

$= (\cancel{1} + \cancel{2i} \cancel{-1}) \cancel{-2} \cancel{-2i} + \cancel{2} = 0$ ✓

- $(1-i)^2 - 2(1-i) + 2$

$= (\cancel{1} - \cancel{2i} \cancel{-1}) \cancel{-2} + \cancel{2i} + \cancel{2} = 0$ ✓

Thus x^2-2x+2 has no real roots but two complex roots: $1 \pm i$

$a + i\,b$

real part — imaginary part.

<u>Definition</u> A <u>Complex number</u> is an entity of the form $a + ib$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

$a+ib = c+id$ if and only if $a = c$ and $b = d$

The set of complex numbers is denoted by $\mathbb{C}$.

Observe that $\mathbb{R} \subseteq \mathbb{C}$ because if $a \in \mathbb{R}$ then $a = a + 0i \in \mathbb{C}$.

It turns out that $\mathbb{C}$ is a <u>field</u>.

The operations of addition and multiplication are easy to define.

<u>Addition</u>

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

multiplication

$$(a+ib)(c+id) = ac + aid + ibc + ibid$$

$$= ac + (ad+bd)i - bd$$

$$= (ac-bd) + (ad+bc)i$$

Division To do this, we need an auxiliary concept. Let $z = a+ib$.

Define $\bar{z} = a-ib$, called the complex conjugate of z. Now observe

$$z\bar{z} = (a+ib)(a-ib)$$

$$= a^2 - \cancel{aib} + \cancel{iba} - i^2 b^2$$

$$= a^2 + b^2 \geq 0 \text{ and real !}$$

Now let $a + ib \neq 0$

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)}$$

$$= \frac{1}{a^2+b^2}\left(a - ib\right)$$

So, to divide by $a+ib$ multiply by $\frac{1}{a+ib}$.

We can think of complex numbers geometrically using the Complex plane