Lecture 11

<u>Definition</u> Let $z = a+ib$.

Define $|z| = +\sqrt{a^2+b^2}$ — a real number

called the <u>modulus of z</u>.

<u>Lemma</u> $|uv| = |u||v|$.

<u>Proof.</u> Let $u = a+ib$ and $v = c+id$.

Then $uv = (ac-bd) + (ad+bc)i$

$$\therefore |uv| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$$

$$= \sqrt{(ac)^2 - 2\cancel{(ac)(bd)} + (bd)^2 + (ad)^2 + 2\cancel{(ad)(bc)} + (bc)^2}$$

$$= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$$

$$|u||v| = \sqrt{a^2+b^2}\sqrt{c^2+d^2}$$

$$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$$

$$= \sqrt{(ac)^2 + (ad)^2 + (bc)^2 + (bd)^2}$$

$\therefore$ $|uv| = |u||v|$ as claimed. $\blacksquare$

The first hint that complex numbers ~~are~~ contain hidden depths is the following result.

<u>Theorem</u> Every non-zero complex number $a+ib$ has exactly two square roots.

<u>Proof</u> Let $(x+iy)^2 = a+ib$.

Then $(x^2-y^2) + 2xyi = a+ib$.

Equating real and imaginary parts

(1) $x^2 - y^2 = a$.

(2) $2xy = b$.

We can obtain a 3rd equation by using moduli

$$|(x+iy)^2| = |a+ib|$$

$$\parallel$$

$$|x+iy|^2$$

Makes sense since $a^2+b^2 \in \mathbb{R}$ and ≥ 0

$\therefore$ (3) $x^2+y^2 = \sqrt{a^2+b^2}$

(1) + (3) gives

$$2x^2 = a + \sqrt{a^2+b^2}$$

We can therefore find two values for x and for each of those values we can find a corresponding value of y. So we can find two values of $x+iy$ (which can be checked to square to $a+ib$) ∎

<u>Example</u> We find the two square roots of $3+4i$.

Let $(x+iy)^2 = 3+4i$

$$\left.\begin{aligned} &(1)\quad x^2-y^2 = 3. \\ &(2)\quad 2xy = 4 \\ &(3)\quad x^2+y^2 = \sqrt{9+16} = 5 \end{aligned}\right\}$$ get 3 equations.

(1) + (3) give

$$2x^2 = 8$$

$\therefore x^2 = 4$

so $\underline{x = \pm 2}$

Now use (2)

$$xy = 2$$

If $x = 2$ then $y = 1$

If $x = -2$ then $y = -1$

$\therefore$ the two square roots of $3+4i$

are $\pm(2+i)$

<u>Check</u>

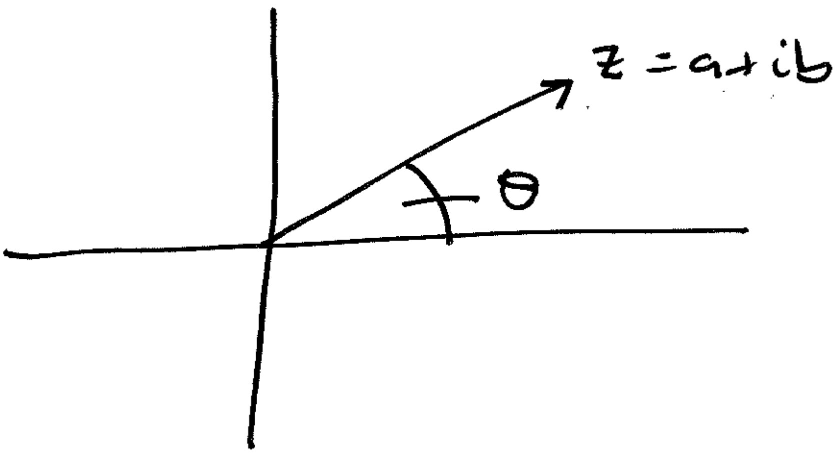
$$(2+i)^2 = 4 + 4i - 1$$
$$= 3+4i \checkmark$$

Remark Let ax^2+bx+c $(a\neq 0)$

be a quadratic polynomial where $a,b,c\in\mathbb{C}$.

Then it always has two roots.

Complex numbers are very geometric.



Let $z \neq 0$.

Then $z = |z|(\cos\theta + i\sin\theta)$

by basic trig. This is called the polar form for complex numbers.

The angle θ (or, more accurately $\theta + 2\pi n$ where $n \in \mathbb{Z}$) is called the argument of the complex number.