Lecture 12

Multiplication of complex numbers has a natural interpretation when viewed geometrically.

<u>Proposition</u> Let $u = |u| (\cos\theta + i \sin\theta)$ and $v = |v| (\cos\phi + i \sin\phi)$. Then

$$uv = |u||v| = (\cos(\theta+\varphi) + i \sin(\theta+\varphi))$$

i.e. multiply the moduli and add the arguments.

<u>Proof</u> $uv = |u| (\cos\theta + i\sin\theta) |v| (\cos\phi + i\sin\phi)$

$$= |u||v| \left((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\phi\cos\theta + \sin\theta\cos\phi) \right)$$

trig identities

$$= |u||v| (\cos(\theta+\varphi) + i \sin(\theta+\varphi)) \blacksquare$$

Example $i \times (\ )$ rotates every complex number by 90° in an anticlockwise direction about the origin. Thus $i^2 \times (\ )$ which is the rotates every complex number by 180° same as $-1 \times (\ )$. This gives us a picture of the sense in which $i^2 = -1$

Deduce

Let $u = |u| (\cos\theta + i\sin\theta)$

Thus $u^2 = |u|^2 (\cos 2\theta + i \sin 2\theta)$

Thus $u^3 = |u|^3 (\cos 3\theta + i \sin 3\theta)$

. . . . .

$$\boxed{u^n = |u|^n (\cos n\theta + i \sin n\theta)}$$

We deduce the following.

$$\boxed{\begin{array}{l}\underline{\text{De Moivre's Theorem}} \\ (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.\end{array}}$$

This has a useful application in proving trig identities.

<u>Example</u> $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

But $(\cos\theta + i\sin\theta)^2$

$= \cos^2\theta + 2\cos\theta\sin\theta\, i + i^2\sin^2\theta$

$= \cos^2\theta + i\, 2\cos\theta\sin\theta - \sin^2\theta$

$= (\cos^2\theta - \sin^2\theta) + i\, 2\cos\theta\sin\theta$

Now equate real and imaginary parts

$$\boxed{\left\{\begin{array}{l}\cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\cos\theta\sin\theta\end{array}\right.}$$

Euler's formula

Assume that we can write

$$(*) \quad e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Need to find $a_0, a_1, a_2, a_3, \dots$

Put $x=0$ $\quad 1 = a_0$.

Now differentiate both sides of $(*)$

$$e^x = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

Put $x=0$ $\quad 1 = a_1$

Now differentiate both sides of the above equn.

$$e^x = 2a_2 + 3 \cdot 2\, a_3 x + \dots$$

Put $x=0$ $\quad 1 = 2a_2 \therefore a_2 = \frac{1}{2}$

Now differentiate both sides of the above eqn

$$e^x = 3 \cdot 2 \cdot a_3 + \ldots$$

Put $x=0$ $\quad \therefore a_3 = \frac{1}{3!}$

etc. We deduce that (with some hand wavy)

$$\boxed{e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots}$$

We can do exactly the same thing for $\sin x$ and $\cos x$ (Rem x is in radians)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We calculate

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots\cdots$$

i	i^2	i^3	i^4	i^5
i	-1	$-i$	1	i

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}i + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) +$$

$$i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This i links $\sin\theta$, $\cos\theta$ and exp

Put $\theta = \pi$

Euler's Formula

$$\boxed{e^{i\pi} = -1}$$