Lecture 13

Write $f(x) \equiv 0$ if
$f(x)$ is identically zero
i.e. all coefficients are 0

Polynomials

A polynomial of degree n with complex coefficients looks like this

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_n \neq 0$ and $a_0, \dots, a_n \in \mathbb{C}$.

A root of $p(x)$ is a number $r \in \mathbb{C}$ s.t. $p(r) = 0$.

We write $\deg p(x)$ for the degree of $p(x)$

[if $p(x)$ is identically zero then $\deg p(x)$ is not defined].

If $a_n = 1$ the polynomial is said to be monic.

Some terminology

degree = 1	Linear
degree = 2	quadratic
degree = 3	cubic
degree = 4	quartic
degree = 5	quintic.

You know all about quadratic polynomials and their roots. In this section, we shall be interested in the roots of arbitrary polynomials.

<u>Theorem</u> (Not proved here) There are <u>algebraic formulae</u> for the roots of quadratics, cubics, and quartics (see my book if you are interested in deriving such formulae). For quintics and beyond there are <u>no</u> algebraic formulae — a result due to Abel and Galois; V. important in the history of algebra.

The above theorem doesn't say polynomials don't have roots, it simply says that you cannot, in general, find them easily.

Let $f(x)$ and $g(x)$ be polynomials.

If $g(x) = f(x)\, q(x)$ for some polynomial $q(x)$, we say that $f(x)$ <u>divides</u> $g(x)$. We call $q(x)$ the <u>quotient</u>. We write

$f(x) \mid g(x)$ read 'f(x) divides g(x)'.

If $f(x) \nmid g(x)$ then

$$g(x) = f(x)\, q(x) + r(x)$$

$r(x)$ is called the <u>remainder</u>.

We have that $\deg r(x) < \deg f(x)$

This result is called the <u>Remainder theorem</u>.

(We shall not prove this here).

<u>Remark</u> You should know from school how to divide by a linear polynomial. See my book (p181) for general long division.

<u>Notation</u>

$\mathbb{C}[x]$ polys with complex coefficients

$\mathbb{R}[x]$ ——— real ———

$\mathbb{Q}[x]$ ——— rational ———

<u>Lemma</u> Let $f(x), g(x)$ be polys (non-zero)

Then $f(x)g(x) \neq 0$ and

$$\deg f(x)g(x) = \deg f(x) + \deg g(x)$$

__Proof__ Let $f(x) = a_m x^m + \text{stuff}$

$g(x) = b_n x^n + \text{stuff}$

where $a_m, b_n \neq 0$.

Then $f(x)\,g(x) = a_m b_n x^{m+n} + \text{stuff}$. $\blacksquare$

Why are we so interested in the roots of polynomials? Because each root gives rise to a linear factor — as we now show.

__Lemma__ $f(a) = 0 \iff (x-a) \mid f(x)$

($\therefore$ $f(x) = q(x)(x-a)$ for some poly $q(x)$ of degree $= (\deg f(x)) - 1$).

Proof *Easy direction*

S'pose that $(x-a) \mid f(x)$. Then

$$f(x) = q(x)(x-a) \quad \text{some } q(x)$$

So $f(a) = q(a)(a-a) = 0$.

It follows that a is a root of $f(x)$.

Hard direction. S'pose that $f(a) = 0$.

There are two possibilities

(1) $(x-a) \mid f(x)$.

(2) $(x-a) \nmid f(x)$.

We show that (2) cannot happen.

$$f(x) = q(x)(x-a) + r(x)$$

where $\deg r(x) < 1$. It follows that $r(x)$ is a constant. Call it r.

Then $f(x) = q(x)(x-a) + r \leftarrow$ no. $\neq 0$

But $f(a) = q(a)0 + r$

$\quad\; \|$

$\quad\; 0$

$\therefore r = 0$. Contradiction. $\blacksquare$

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Suppose $f(x) = q(x)(x-a)$

Then $\deg f(x) = \deg q(x) + \deg(x-a)$

$= \deg q(x) + 1$

$\therefore \deg q(x) = \deg f(x) - 1$.

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Consequence If a is a root of $f(x)$

then $f(x) = (x-a)\,q(x)$ where

$\deg q(x) = \deg f(x) - 1$.

Thus $q(x)$ is "simpler" than $f(x)$.