Lecture 14

Let $p(x)$ be a non-constant polynomial. Last time, we saw that

$$\boxed{p(a) = 0 \iff (x-a) \mid p(x)}$$

a is a root (a number)

$(x-a)$ is a factor (a linear polynomial)

<u>Remark</u> Let a be a root of $p(x)$. We say that a has <u>multiplicity m</u> if $(x-a)^m \mid p(x)$ but $(x-a)^{m+1} \nmid p(x)$

This gives a precise meaning to the intuitive idea of a root being repeated.

Our first big theorem is the following

<u>Theorem</u> A non-constant polynomial of degree n has <u>at most</u> n roots

<u>Proof</u> (Best proved via induction — see later). If $f(x)$ has a root a_1 then

$\deg f(x) = n$

$$f(x) = (x-a_1)\, f_1(x)$$

where $\deg f_1(x) = n-1$

Now repeat with $f_1(x)$ etc.

The max. no. of roots $f(x)$ can have is n in which case

$$f(x) = \underset{\text{constant}}{a}\,\underbrace{(x-a_1)\ldots(x-a_n)}_{n \text{ roots}}$$

■

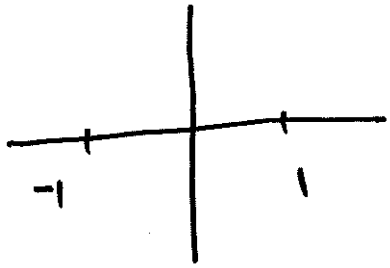
Special case: the polynomial $x^n - a$ $(a \in \mathbb{C},\ a \neq 0)$

The roots of this polynomial are called nth roots of a. We shall prove that there are exactly n of them (we know there are at most n).

The case $a = 1$

The roots of $x^n - 1$ are called nth roots of unity.

<u>Examples</u>

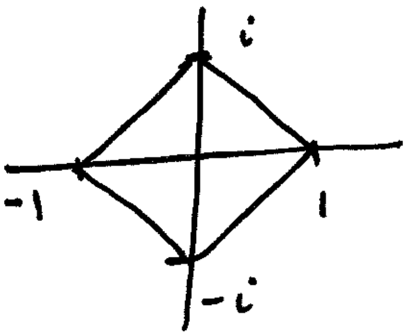


(1) The roots of x^2-1 are ± 1.

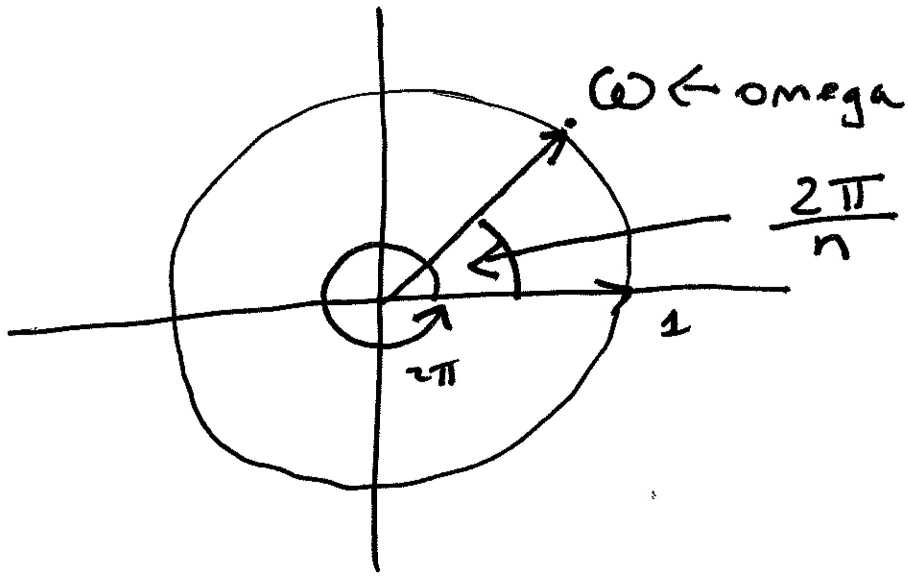
These are the two square roots of unity

(2) The roots of x^4-1 are $\pm 1, \pm i$.

These are the four fourth roots of unity.



To find all the n^{th} roots of unity, we first locate a special n^{th} root (not 1!) called the <u>principal n^{th} root</u>.



$$1 = \cos 2\pi + i \sin 2\pi$$

<u>Define</u> $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$

By properties of complex multiplication or de Moivre's theorem.

$$\omega^n = \cos n \frac{2\pi}{n} + i \sin n \frac{2\pi}{n}$$

$$= \cos 2\pi + i \sin 2\pi$$

$$= 1$$

We call ω <u>the principal n^{th} root of unity</u>

Now consider the following n complex no's

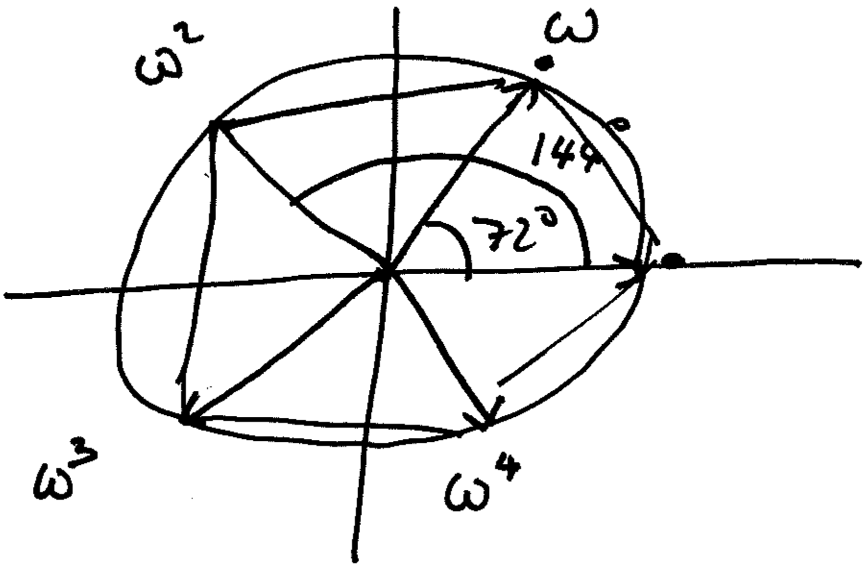
$$\underbrace{\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}, \{\omega^n = 1\}}$$

All n^{th} roots of unity

ω^2 is obtained from ω by doubling the angle of ω

ω^3 ———————— tripling the angle of ω etc.

Example We find the 5th root of unity



$\frac{2\pi}{5} = 72^0$

216

288

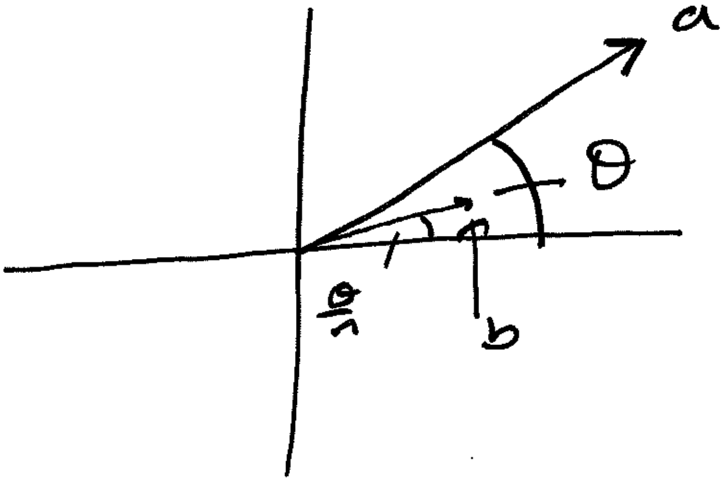
360

$\omega, \omega^2, \omega^3, \omega^4, 1$ form the vertices of a regular pentagon.

We have shown that $x^n - 1$ has exactly n roots. We now look at the general equn. $x^n - a$

(1) Let $\omega, \omega^2, \dots, \omega^{n-1}, 1$ be the n^{th} roots of unity as before.

(2)



$a = r(\cos\theta + i\sin\theta)$ [if $\theta = 0$ replace by 2π]
$r \geq 0$, $r \in \mathbb{R}$

Define $b = \underbrace{\sqrt[n]{r}}_{\text{real } n^{th} \text{ root}} \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

Called the principal n^{th} root of a

$$b^n = r\left(\cos n\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

$$= r$$

The nth roots of a are

$$\underbrace{\omega b,\ \omega^2 b,\ \dots,\ \omega^{n-1} b,\ b}$$

- These are all different (arguments differ)
- There are n of them
- Their n^{th} powers are a.

Theorem The equation $x^n = a$ $(a \neq 0)$ has exactly n roots

$$\omega b,\ \omega^2 b,\ \dots,\ \omega^{n-1} b,\ b$$

where ω is the principal n^{th} root of unity and $b^n = a$ ∎