Lecture 15

Example

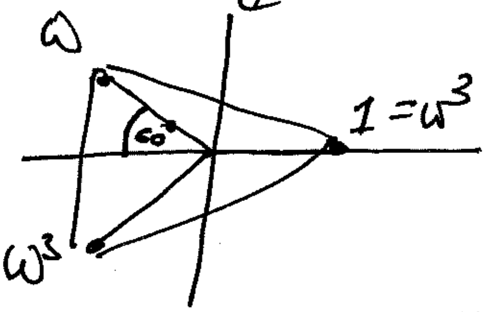
trig form

Third roots of unity.

Principal third root is

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

equilateral Δ



The roots are

$$\omega,\ \omega^2,\ \omega^3 = 1$$

But in this case, we can work out $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$ explicitly

$$\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$$

radical expression

$$\left[\ \omega^2 = -\frac{1}{2}\left(1 + i\sqrt{3}\right),\quad \omega^3 = 1 \text{ (of course)}\ \right]$$

A radical expression is (roughly speaking) an algebraic expression involving roots (no sines or cosines!)

The following will not be proved

Theorem (The fundamental theorem of algebra)

Every non-constant poly has a root

Corollary Let $p(x)$ be a poly of degree n.

Then $p(x)$ has exactly n roots and

$$p(x) = a(x-a_1)\cdots(x-a_n)$$

$a \in \mathbb{C}$.

$\mathbb{C}[x]$ polys with complex coeffs.

$\mathbb{R}[x]$ polys with real coeffs.

We now apply the above results to real polynomials:

Properties of Complex Conjugates

Lemma

1. $\overline{z_1 + \dots + z_n} = \overline{z_1} + \dots + \overline{z_n}$
2. $\overline{z_1 \cdot \dots \cdot z_n} = \overline{z_1} \cdot \dots \cdot \overline{z_n}$
3. z is real iff $\overline{z} = z$.

[See page 167].

Lemma Let $p(x) \in \mathbb{R}[x]$.

If z is a root then $\overline{z}$ is a root.

Proof. $p(x) = a_n x^n + \dots + a_1 x + a_0$.

$$a_n z^n + \dots + a_1 z + a_0 = 0$$

$$\overline{a_n z^n + \dots + a_1 z + a_0} = \overline{0}$$

$$\dots\dots$$

$$\overline{a_n}\,\overline{z}^n + \dots + \overline{a_1}\,\overline{z} + \overline{a_0} = 0$$

$$a_n \overline{z}^n + \dots + \overline{a_1}\,\overline{z} + a_0 = 0.$$

So, $p(\overline{z}) = 0$ ∎

"Buy one root, get one root for free".

<u>Lemma</u> Let $z \in \mathbb{C} \setminus \mathbb{R}$.
Then $(x-z)(x-\overline{z})$ is a real irreducible quadratic polynomial.

<u>Proof.</u> $(x-z)(x-\overline{z}) = x^2 - x\overline{z} - zx + z\overline{z}$

$= x^2 - x(\overline{z}+z) + z\overline{z}$

Let $z = a+ib$. Then $z\overline{z} = a^2+b^2$

$\overline{z}+z = (a-ib)+(a+ib) = 2a$.

$\therefore\ x^2 - \underline{2a}\,x + \underline{a^2+b^2} \in \mathbb{R}[x]$

$D = 4a^2 - 4(a^2+b^2) = -4b^2 < 0$ ∎

<u>Theorem</u> (Fundamental theorem of real polys)

Every real polynomial can be written as a product of <u>real linear</u> and <u>real irreducible quadratic poly</u>

<u>Proof</u> let $p(x) \in \mathbb{R}[x]$. ~~y~~

Divide the roots into real $r_1, \ldots, r_s$ and complex $z_1, \ldots z_t$. But, the complex roots come in conjugate pairs $u_1, \overline{u_1}, u_2, \overline{u_2}, \ldots, u_m, \overline{u_m}$

$$\therefore p(x) = a(x-r_1)\ldots(x-r_s)\overbrace{(x-u_1)(x-\overline{u_1})}\ldots(x-u_m)(x-\overline{u_m})$$

real const.

$(x-u_i)(x-\overline{u_i})$ multiplies out to give a real irreducible quadratic ∎

How to find roots?

Not many options at the moment

(1) If $p(x) \in \mathbb{R}[x]$ and you are given a root $u \in \mathbb{C} \setminus \mathbb{R}$ then $\bar{u}$ will also be a root.

(2) If $p(x)$ has integer coefficients any integer roots must divide the constant term of $p(x)$.

Example $x^4 - 3x^3 + 3x^2 - 3x + 2$

$p(x) = \cancel{x^4 - 3x^3 + 3x^2 - 3x + 2}$

Try $\pm 1, \pm 2$

Find that 1, and 2 are roots

calculated by division

$$\boxed{p(x) = (x-1)(x-2)(x^2+1)}$$

x^2+1 is irreducible quadratic since $D < 0$.

$$[(x-1)(x-2) = x^2 - 3x + 2]$$