Lecture 16

SECTION 3:

Matrices The goal of this section is to develop a systematic procedure for solving systems of linear equations.

Definition A *matrix* (pl. matrices) is a rectangular array of numbers (also other mathematical quantities).

columns

rows
$$\begin{pmatrix} 3 & 4 & 6 & 7 \\ 8 & 0 & -\pi & -e \\ \sqrt{2} & 1 & i & -i \end{pmatrix} \leftarrow \text{a } 3\times 4 \text{ matrix}$$

$m \times n$ matrix m is the # of *rows* and n is the number of *columns*. Called the *size* of the matrix

Matrices are usually denoted by uppercase Latin letters $A, B, C, \ldots$

$(A)_{ij}$ = the element in the i^{th} row and j^{th} column.

<u>Example</u>

If $A = \begin{array}{cc} \begin{array}{cccc} & & 3 & \end{array} & \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} & \begin{array}{c} \\ \leftarrow 2 \end{array} \end{array}$

$(A)_{23} = 7$

<u>Definition</u> ~~Two~~ Matrices A and B are equal if they have the <u>same size</u> and <u>corresponding elements are equal</u> i.e

$$(A)_{ij} = (B)_{ij}$$

for all appropriate i and j

Matrix arithmetic

Addition Matrices A and B can only be added if they have the same size. Then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

Subtraction If A and B have the same size

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij}$$

Scalar multiplication In matrix theory, numbers are called scalars (denoted by Greek letters $\lambda, \mu, \nu, \ldots$)

$$(\lambda A)_{ij} = \lambda (A)_{ij}$$

The transpose of A

$$(A^T)_{ij} = (A)_{ji}$$

Example $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$

Row matrix

$$(a_1 \ a_2 \ a_3 \ldots a_m)$$

Column matrix

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

The inner product of a row matrix and a column matrix

$\underline{a} = (a_1 \ldots a_m)$

$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

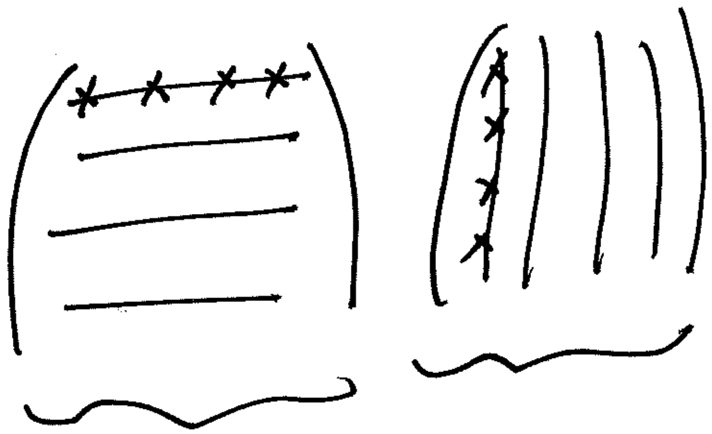
$\underline{a} \cdot \underline{b}$ is only defined if $m = n$

in which case

$\underline{a} \cdot b = \underbrace{a_1 b_1 + \ldots + a_m b_m}_{\text{a single number.}}$

Matrix multiplication This is more complex than the other matrix operations, but very important. Want to define

$$AB$$



think of as rows   think of as columns.

Need

columns of A = # rows of B

(else NOT DEFINED)

$$A \quad \begin{pmatrix} \leftarrow \underline{a}_1 \rightarrow \\ \vdots \\ \leftarrow \underline{a}_m \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & & \uparrow \\ \underline{b}_1 & \cdots & \underline{b}_n \\ \downarrow & & \downarrow \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}_1 \cdot \underline{b}_1 & \underline{a}_1 \cdot \underline{b}_2 & \cdots & \underline{a}_1 \cdot \underline{b}_n \\ \vdots & & & \\ \underline{a}_m \cdot \underline{b}_1 & \cdots & \cdots & \underline{a}_m \cdot \underline{b}_n \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$2 \times 3 \qquad 3 \times 2$

$$= \begin{pmatrix} (1\ 2\ 1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & (1\ 2\ 1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ (0\ 1\ -1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & (0\ 1\ -1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

If A is $m \times p$ and B is $p \times n$

then AB is $m \times n$

As usual $A^2 = AA$

$A^3 = AAA$

if defined