Lecture 17

Convention $(A)_{ij} = a_{ij}$

i^{th} row $\qquad\qquad\qquad\qquad$ j^{th} column

$$\begin{pmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{pmatrix} \begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{pmatrix} =$$

$A \qquad\qquad\qquad B$

$m \times n \qquad\qquad n \times p$

$$\begin{array}{c|c} & j \\ \hline i & \sum_{k=1}^{n} a_{ik} b_{kj} \end{array}$$

$AB \qquad m \times p$

As usual A^2 means AA

$A^3 = AA^2$ etc

Special matrices.

\# rows = # columns   square matrix

all entries zero   zero matrix O

Identity matrices $\underset{I_1}{(1)}, \underset{I_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{I_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \ldots$ etc

usually just write I and allow context to determine size.

$A \underline{x} = \underline{b}$ is a system of linear equations in matrix form.

A - matrix of coefficients

$\underline{x}$ - unknown (column vector)

$\underline{b}$ - given (column vector)

Any $\underline{a}$ s.t. $A \underline{a} = \underline{b}$ is called a <u>solution</u>. The <u>solution set</u> of $A \underline{x} = \underline{b}$ is the set

$$\{ \underline{a} : A \underline{a} = \underline{b} \}.$$

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Properties of matrices

This is a version of Section 8.2.

Properties of matrix addition

We restrict attention to the set of all $m \times n$ matrices.

(MA1): $(A+B)+C = A+(B+C)$. This is the *associative law* for matrix addition.

(MA2): $A+O = A = O+A$. The zero matrix O, the same size as A, is the *additive identity* for matrices the same size as A.

(MA3): $A+(-A) = O = (-A)+A$. The matrix $-A$ is the unique *additive inverse* of A.

(MA4): $A+B = B+A$. Matrix addition is *commutative.*

Properties of matrix multiplication

(MM1): The product $(AB)C$ is defined precisely when the product $A(BC)$ is defined, and when they are both defined $(AB)C = A(BC)$. This is the *associative law* for matrix multiplication.

(MM2): Let A be an $m \times n$ matrix. Then $I_m A = A = AI_n$. The matrices I_m and I_n are the *left and right multiplicative identities*, respectively. It is important to observe that for matrices that are not square different identities are needed on the left and on the right.

(MM3): $A(B+C) = AB+AC$ and $(B+C)A = BA+CA$ when the products and sums are defined. These are the *left and right distributivity laws*, respectively, for matrix multiplication over matrix addition.

Properties of scalar multiplication

(S1): $1A = A$ and $-1A = -A$.

(S2): $0A = O$.

(S3): $\lambda(A+B) = \lambda A + \lambda B$ where λ is a scalar.

(S4): $(\lambda\mu)A = \lambda(\mu A)$ where λ and μ are scalars.

(S5): $(\lambda+\mu)A = \lambda A + \mu A$ where λ and μ are scalars.

(S6): $(\lambda A)B = A(\lambda B) = \lambda(AB)$ where λ is a scalar.

Properties of the transpose

(T1): $(A^T)^T = A$.

(T2): $(A+B)^T = A^T + B^T$.

(T3): $(\lambda A)^T = \lambda A^T$ where λ is a scalar.

(T4): $(AB)^T = B^T A^T$. **Pay close attention to this property.**

Counterexamples

(1) *Matrix multiplication is not commutative.* For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

then $AB \neq BA$. One consequence of this is that

$$(A+B)^2 \neq A^2 + 2AB + B^2,$$

in general.

(2) *The product of two matrices can be a zero matrix without either matrix being a zero matrix.* For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$$

then $AB = O$.

(3) *Cancellation of matrices is not allowed in general.* For example, if

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$$

then $A \neq O$ and $AB = AC$ but $B \neq C$.

5(a)

(1)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 7 \end{pmatrix}$$

$$\neq$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(3)

$$AB = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$$

$$\parallel$$

$$AC = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$$

All of these properties of matrices can be proved from the definitions. I will give only one example to provide a flavour.

Proof of associativity of matrix multiplication

Need to prove that

$$(AB)\,C = A(BC)$$

(when defined).

(1) Need to show that $(AB)C$ and $A(BC)$ have the same size.

$$\underset{m \times p}{A} \quad \underset{p \times n}{B} \quad \underset{n \times q}{C}$$

The size of $(AB)C$ is

$$(m \times n)(n \times q) = \underline{\underline{m \times q}}$$

The size of $A(BC)$ is

$$(m \times p)(p \times q) = \underline{\underline{m \times q}}$$

It follows that $(AB)C$ and $A(BC)$ do have the same size. We now prove that corresponding entries are equal. We calculate

$$((AB)C)_{ij} \quad \text{and} \quad (A(BC))_{ij}$$

$$((AB)C)_{ij} = i\text{th row of } (AB) \cdot j^{\text{th}} \text{ column of } C$$

$$= \sum_k (AB)_{ik} (C)_{kj}$$

$$(AB)_{ik} = i\text{th row of } A \bullet k^{\text{th}} \text{ column of } B$$

$$= \sum_l (A)_{il} \cancel{(B)} (B)_{lk}$$

$$\therefore \left((AB)C\right)_{ij} =$$

distributivity of numbers.

$$\sum_k \left(\sum_l (A)_{il} (B)_{lk} \right) (C)_{kj}$$

$$= \underline{\underline{\sum_k \sum_l (A)_{il} (B)_{lk} (C)_{kj}}} .$$

$\left(A(BC)\right)_{ij}$ = i^{th} row of A $\cdot$ j^{th} column of (BC)

$$= \sum_l (A)_{il} (BC)_{lj}$$

$$= \sum_l (A)_{il} \left(\sum_k (B)_{lk} (C)_{kj} \right)$$

$$= \underline{\sum_l \sum_k (A)_{il} (B)_{lk} (C)_{kj}}$$

We are adding up the same numbers in a different order and so their sums must be the same. ∎