Lecture 18

Solving systems of linear equations

We use the matrix properties discussed in the last lecture.

Theorem (Fundamental theorem of linear equations)

Let $A\underline{x} = \underline{b}$ be a system of linear equations (over $\mathbb{C}$). Then there are 3 possibilities.

(1) There are no solutions. } Inconsistent.

(2) Exactly one solution.

(3) Infinitely many solutions. } Consistent

Proof Assume that there are at least two solutions $\underline{u}$ and $\underline{v}$ ($\underline{u} \neq \underline{v}$). So, $A\underline{u} = \underline{b}$ and $A\underline{v} = \underline{b}$.

Then $A(\underline{u}-\underline{v}) = A\underline{u} - A\underline{v} = \underline{b} - \underline{b} = \underline{0}$

Let λ be any scalar. Then

$$A\left(\lambda(\underline{u}-\underline{v})+\underline{u}\right) = \lambda A(\underline{u}-\underline{v}) + A\underline{u}$$
$$= \underline{0} + \underline{b} = \underline{b}$$

$\therefore$ for all $\lambda \in \mathbb{C}$,

$\lambda(\underline{u}-\underline{v}) + \underline{u}$

are solutions. We have therefore found infinitely many solutions. $\blacksquare$

System of linear equations

inconsistent?

Consistent?

unique solution?

infinitely many solutions?

We shall now develop ~~approach~~ an algorithm to answer these questions.

Let $A\underline{x} = \underline{b}$ be our system of linear equations. We will operate with the <u>augmented matrix</u> $(A|\underline{b})$. We shall apply what are known as <u>elementary row operations</u> (EROs) to the augmented matrix $(A|\underline{b})$. This will yield a new augmented matrix matrix $(A'|\underline{b}')$ which has a shape making it an <u>echelon matrix</u>.

The equations associated with $(A'|\underline{b}')$ are

(1) Easy to solve

(2) Have the same solutions as the original set of equations

$$A\underline{x} = \underline{b}$$

$$\downarrow$$

$$(A|\underline{b}) \xrightarrow{\text{ERO's}} \underset{\text{echelon matrix}}{(A'|\underline{b}')} \nearrow A'\underline{x} = \underline{b}'$$

easy to solve & have same solutions as $A\underline{x} = \underline{b}$

Elementary row operations

1. $R_i \leftarrow \lambda R_i \quad (\lambda \neq 0)$

   multiply row i by a non-zero scalar.

2. $R_i \leftrightarrow R_j$.

   Interchange row i and row j

3. $R_j \leftarrow R_j + \lambda R_i$

   Add λ times row i to row j

Proposition. Let $A\underline{x} = \underline{b}$ be a system of linear equations. Apply ERO to $(A|\underline{b})$ to obtain $(A'|\underline{b}')$. Then the solution set of $A\underline{x} = \underline{b}$ is the same as the solution set of $A'\underline{x} = \underline{b}'$.

$$\boxed{\text{The EROs change the equations BUT do not change the solutions}}$$

Example

system of linear equations.

$$\begin{cases} x + 2y - 3z = -1 \\ 3x - y + 2z = 7 \\ 5x + 3y - 4z = 2 \end{cases}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

$R_2 \leftarrow R_2 - 3R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

Calculations

$$\begin{array}{cccc} -3 & -6 & 9 & 3 \\ 3 & -1 & 2 & 7 \\ \hline 0 & -7 & 11 & 10 \end{array}$$

$R_3 \leftarrow R_3 - 5R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right)$$

$R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$$\begin{array}{cccc} -5 & -10 & 15 & 5 \\ 5 & 3 & -4 & 2 \\ \hline 0 & -7 & 11 & 7 \end{array}$$

The equations associated with this augmented matrix are

$$x + 2y - 3z = -1$$

$$-7y + 11z = 10$$

$$\boxed{0x + 0y + 0z = -3}$$

$0 = -3$ # NO SOLUTIONS

It follows that the original system of equations is inconsistent. $\square$

We shall next describe echelon matrices and then describe the algorithm for solving systems of linear equations.