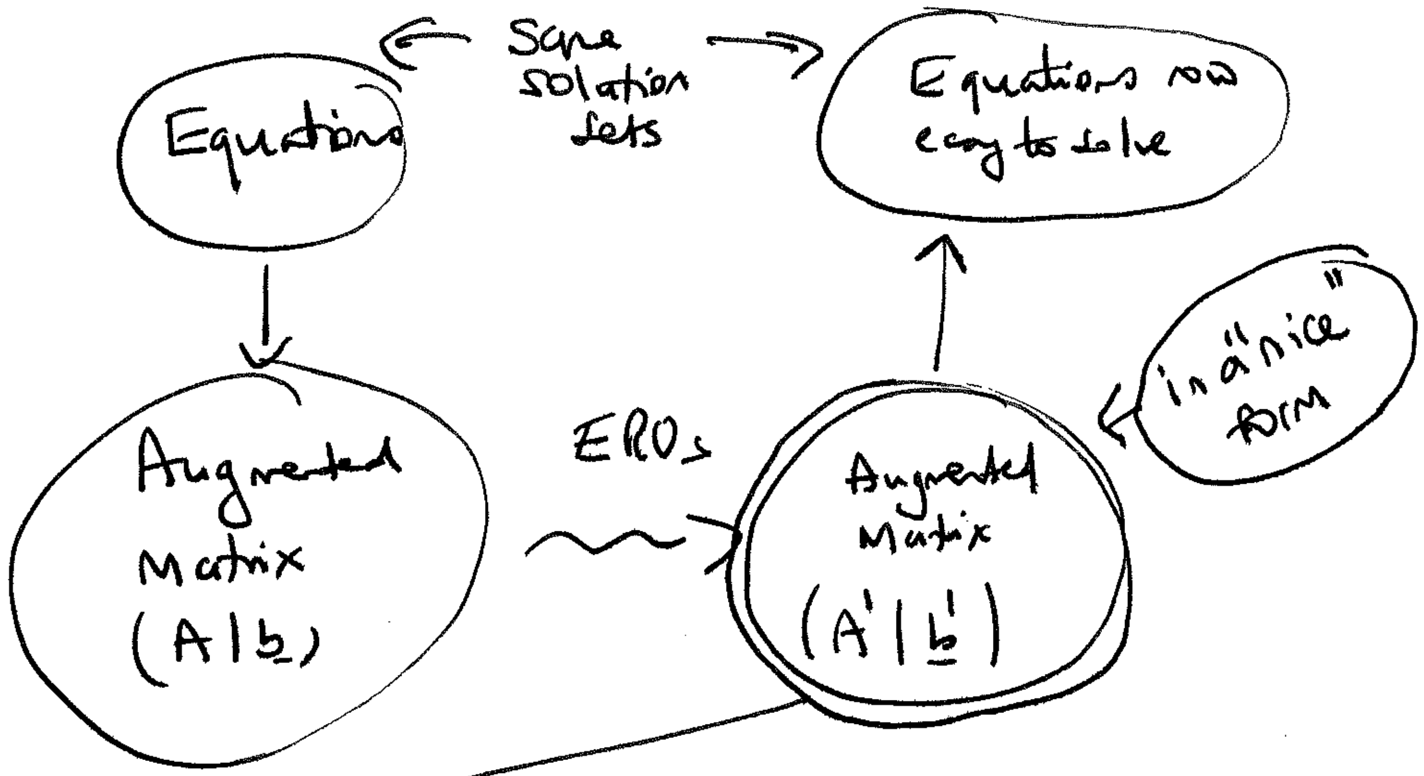
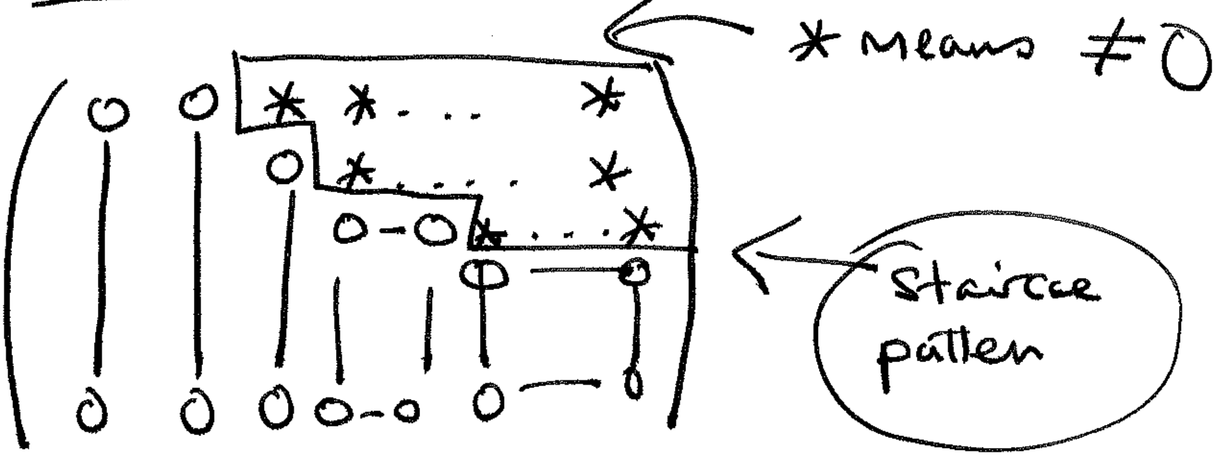
Lecture 19 | Using EROs to solve systems of linear equations continued...



- The matrix $(A'|\underline{b}')$ has to be an <u>echelon matrix</u>. These look like this



Special case when A is 3×3.

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right)$$

Use top left non-zero entry to produce 0s below it using EROs

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right)$$

Now repeat with leading non-zero entry in next row.

Various patterns can arise:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right)$$ ← inconsistent

We have already seen an example of this.

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

Consistent

unique solution

(I will show you an example in a minute)

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$$ ← consistent infinitely many solutions.

or

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Examples

These illustrate the three kinds of outcome with the Fundamental Theorem of Linear Equations.

We did this example in lecture 18

1. We show that the following system is Inconsistent.

$$x + 2y - 3z = -1$$
$$3x - y + 2z = 7$$
$$5x + 3y - 4z = 2$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$$0x + 0y + 0z = 3$$

$$0 = -3$$

inconsistent

$R_2 \leftarrow R_2 - 3R_1$

-3	-6	9	3
3	-1	2	7
0	-7	11	10

-5	-10	15	5
5	3	-4	2
0	-7	11	7

$R_3 \leftarrow R_3 - 5R_1$

$R_3 \leftarrow R_3 - R_2$

staircase pattern

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ \underbrace{0 \quad 0 \quad 0}_{\text{all 0s}} & & & \underbrace{-3}_{\text{non-zero}} \end{array}\right)$$

inconsistent.

2. We show that the following system has exactly one solution

System of equations.

$$x + 2y + 3z = 4$$

$$2x + 2y + 4z = 0$$

$$3x + 4y + 5z = 2$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 3 & 4 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & -2 & -4 & -10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Unique solution

$R_2 \leftarrow R_2 - 2R_1$

$$\begin{array}{cccc} -2 & -4 & -6 & -8 \\ 2 & 2 & 4 & 0 \\ \hline 0 & -2 & -2 & -8 \end{array}$$

$R_3 \leftarrow R_3 - 3R_1$

$$\begin{array}{cccc} -3 & -6 & -9 & -12 \\ 3 & 4 & 5 & 2 \\ \hline 0 & -2 & -4 & -10 \end{array}$$

$R_3 \leftarrow R_3 - R_2$

$$\begin{array}{cccc} 0 & -2 & -4 & -10 \\ 0 & 2 & 2 & 8 \\ \hline 0 & 0 & -2 & -2 \end{array}$$

$R_2 \leftarrow -\frac{1}{2} R_2$

$R_2 \leftarrow -\frac{1}{2} R_3$

$$x + 2y + 3z = 4$$
$$y + z = 4$$
$$z = 1$$

back substitution.

$$y = 4 - z = 4 - 1 = 3$$

$$x = 4 - 2y - 3z$$
$$= 4 - 6 - 3 = -5$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$

<u>Check</u>

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -5+6+3 = 4 \\ -10+6+4 = 0 \\ -15+12+5 = 2 \end{pmatrix}$$

This is how you check answers.

3. We solve a system where there are infinitely many solutions.

$$\left.\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}\right\} \text{ equations.}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right)$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array}\right)$$

$\downarrow$ $R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$R_2 \leftarrow R_2 - 2R_1$

-2	-4	6	-12
2	-1	4	2
0	-5	10	-10

$R_3 \leftarrow R_3 - 4R_1$

-4	-8	12	-24
4	3	-2	14
0	-5	10	-10

$R_2 \leftarrow -\frac{1}{5} R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ NMOS → Consistent.

$$x + 2y - 3z = 6$$
$$y - 2z = 2$$

One free variable

Put $z = \lambda$ any scalar $\quad y = 2 + 2\lambda$

$$x = 6 - 2y + 3z$$
$$= 6 - 2(2 + 2\lambda) + 3\lambda$$
$$= 6 - 4 - 4\lambda + 3\lambda$$
$$= 2 - \lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - \lambda \\ 2 + 2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Check

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2-\lambda \\ 2+2\lambda \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2 - \underline{\lambda} + \underline{4} + \lambda 4 - 3\lambda = 6 \\ 4 - \cancel{2\lambda} - 2 - \cancel{2\lambda} + \cancel{4\lambda} = 2 \\ 8 - \cancel{4\lambda} + 6 + \cancel{6\lambda} - \cancel{2\lambda} = 14 \end{pmatrix}$$

This is how you check solutions.

Example A system such as

$$x + 2y + 3z = 4$$

will have two free variables. Put $z = \lambda$ and $y = \mu$

Then $x = 4 - 2\mu - 3\lambda$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\mu - 3\lambda \\ \mu \\ \lambda \end{pmatrix}$ is the general solution where $\lambda, \mu \in \mathbb{C}$