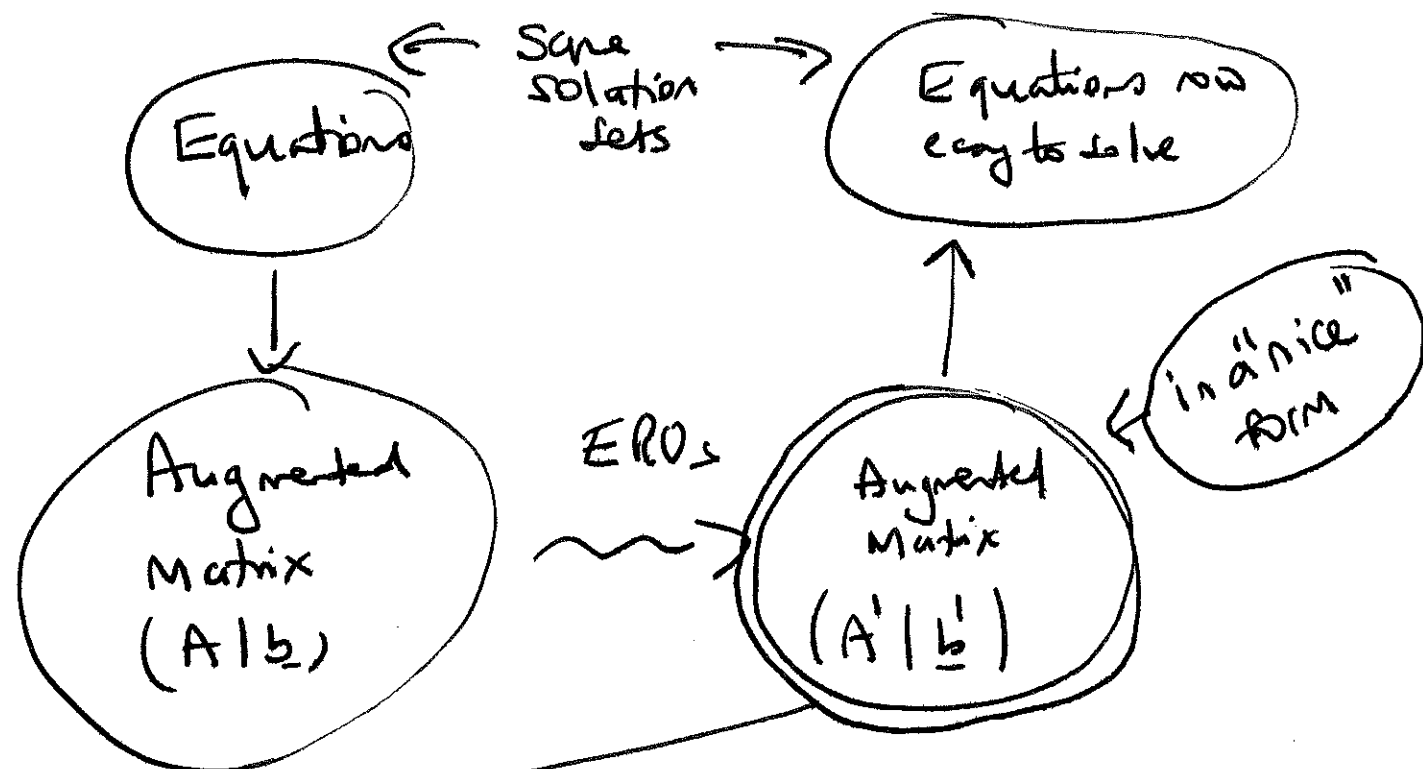
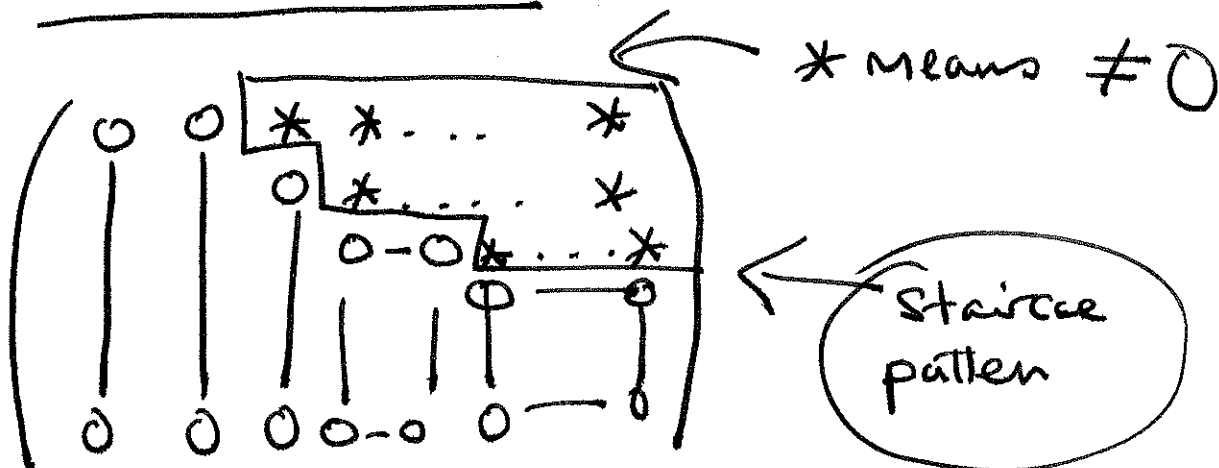


Lecture 19

Using EROs to solve systems of linear equations continued...



The matrix $(A'|b')$ has to be an echelon matrix. These look like this



Special case When A is 3×3 .

$$\begin{pmatrix} \textcircled{*} & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix}$$

Diagram showing the first row of a 3×3 matrix with the top-left element circled. Arrows indicate the elimination process: from the circled element to the other elements in the first column, and from the other elements in the first row to the other elements in the first column.

Use top left non-zero entry to produce 0s below it using EROs

$$\begin{pmatrix} * & * & * & | & * \\ 0 & \textcircled{*} & * & | & * \\ 0 & * & * & | & * \end{pmatrix}$$

Diagram showing the result of the first step: the first column has zeros below the top-left element, which is now circled. Arrows indicate the elimination process: from the circled element to the other elements in the second column, and from the other elements in the second column to the other elements in the second column.

Now repeat with leading non-zero entry in next row.

Various patterns can arise:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array} \right) \leftarrow \text{inconsistent}$$

We have already seen an example of this.

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array} \right) \begin{array}{l} \text{Consistent} \\ \hline \text{Unique solution} \end{array}$$

(I will show you an example in a minute)

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{array} \right) \leftarrow \begin{array}{l} \text{Consistent} \\ \text{infinitely many} \\ \text{solutions} \end{array}$$

or

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \leftarrow$$

Examples

These illustrate the three kinds of outcome
in the Fundamental Theorem of Linear Equations.

We did this example in lecture 18

1. We show that the following system is
Inconsistent.

$$x + 2y - 3z = -1$$

$$3x - y + 2z = 7$$

$$5x + 3y - 4z = 2$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array} \right)$$

5

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 5 & 3 & -4 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array} \right)$$

$$0x + 0y + 0z = 3$$

$$0 = -3$$

inconsistent

$$R_2 \leftarrow R_2 - 3R_1$$

$$\begin{array}{cccc} -3 & -6 & 9 & 3 \\ 3 & -1 & 2 & 7 \\ \hline 0 & -7 & 11 & 10 \end{array}$$

$$\begin{array}{cccc} -5 & -10 & 15 & 5 \\ 5 & 3 & -4 & 2 \\ \hline 0 & -7 & 11 & 7 \end{array}$$

$$R_3 \leftarrow R_3 - 5R_1$$

$$R_3 \leftarrow R_3 - R_2$$

← staircase pattern

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array} \right)$$

all 0s

non-zero

inconsistent.

2. We show that the given system
has exactly one solution

System of equations:

$$x + 2y + 3z = 4$$

$$2x + 2y + 4z = 0$$

$$3x + 4y + 5z = 2$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array} \right)$$

7

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 3 & 4 & 5 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & -2 & -4 & -10 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & -2 & -2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

Unique solution

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{array}{cccc} -2 & -4 & -6 & -8 \\ 2 & 2 & 4 & 0 \\ \hline 0 & -2 & -2 & -8 \end{array}$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$\begin{array}{cccc} -3 & -6 & -9 & -12 \\ 3 & 4 & 5 & 2 \\ \hline 0 & -2 & -4 & -10 \end{array}$$

$$R_3 \leftarrow R_3 - R_2$$

$$\begin{array}{cccc} 0 & -2 & -4 & -10 \\ 0 & 2 & 2 & 8 \\ \hline 0 & 0 & -2 & -2 \end{array}$$

$$R_2 \leftarrow -\frac{1}{2}R_2$$

$$R_2 \leftarrow -\frac{1}{2}R_3$$

8

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + z &= 4 \\ z &= 1 \end{aligned} \quad \left. \begin{array}{l} \nearrow \\ \nearrow \\ \nearrow \end{array} \right\} \text{back substitute.}$$

$$y = 4 - z = 4 - 1 = 3$$

$$\begin{aligned} x &= 4 - 2y - 3z \\ &= 4 - 6 - 3 = -5 \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$

Check

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 + 6 + 3 = 4 \\ -10 + 6 + 4 = 0 \\ -15 + 12 + 5 = 2 \end{pmatrix}$$

This is how you check answers.

3. We solve a system where there are

infinitely many solutions.

$$\left. \begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned} \right\} \text{equations.}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array} \right)$$

↓

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array} \right)$$

↓

$$R_3 \leftarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{array}{cccc} -2 & -4 & 6 & -12 \\ 2 & -1 & 4 & 2 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 4 & 2 \end{array}$$

$$\hline \begin{array}{cccc} 0 & -5 & 10 & -10 \end{array}$$

$$R_2 \leftarrow R_2 - 4R_1$$

$$\begin{array}{cccc} -4 & -8 & 12 & -24 \\ 4 & 3 & -2 & 14 \end{array}$$

$$\begin{array}{cccc} 4 & 3 & -2 & 14 \end{array}$$

$$\hline \begin{array}{cccc} 0 & -5 & 10 & -10 \end{array}$$

$$R_1 \leftarrow -\frac{1}{8}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ MOS is consistent.}$$

$$x + 2y - 3z = 6$$

$$y - 2z = 2$$

one free variable

Put $z = \lambda$ any scalar

$$y = 2 + 2\lambda$$

$$x = 6 - 2y + 3z$$

$$= 6 - 2(2 + 2\lambda) + 3\lambda$$

$$= 6 - 4 - 4\lambda + 3\lambda$$

$$= 2 - \lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - \lambda \\ 2 + 2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Check

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 2-\lambda \\ 2 & -1 & 4 & 2+2\lambda \\ 4 & 3 & -2 & \lambda \end{array} \right)$$

$$= \left(\begin{array}{l} 2 - \lambda + 4 + \lambda 4 - 3\lambda = 6 \\ 4 - 2\lambda - 2 - 2\lambda + 4\lambda = 2 \\ 8 - 4\lambda + 6 + 6\lambda - 2\lambda = 14 \end{array} \right)$$

This is how you check solutions.

Example A system such as

$$x + 2y + 3z = 4$$

will have two free variables. Put $z = \lambda$ and $y = \mu$

$$\text{Then } x = 4 - 2\mu - 3\lambda$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\mu - 3\lambda \\ \mu \\ \lambda \end{pmatrix} \text{ is the general solution} \\ \text{where } \lambda, \mu \in \mathbb{C}$$